Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 10, 2011

1. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = x^3$, for $0 \le x \le 1$, about the x-axis.

(b) Do the same for the curve $y = |x|$, for $-1 \le x \le 1$.

Solution:

(a)

$$\begin{aligned}
A &= 2\pi \int_0^1 y\sqrt{1+(dy/dx)^2}\,dx \\
&= 2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx \qquad\qquad (u = 1+9x^4,\ du = 36x^3\,dx) \\
&= \frac{2\pi}{36}\int_1^{10} \sqrt{u}\,du \\
&= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} \;=\; \frac{\pi}{27}\left(10^{3/2}-1\right) \;=\; \frac{\pi}{27}\left(10\sqrt{10}-1\right)
\end{aligned}$$

(b) We have

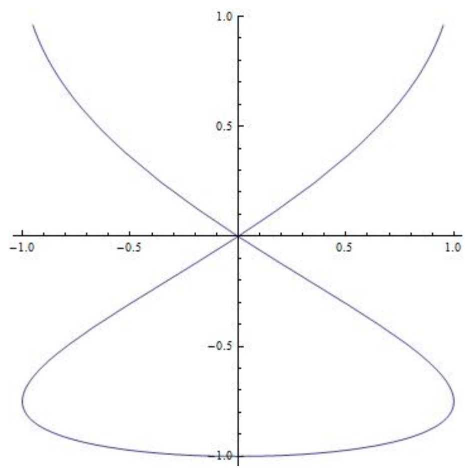
$$\frac{dy}{dx} = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

so

$$\begin{aligned}
A &= 2\pi \int_{-1}^0 (-x)\sqrt{1+(-1)^2}\,dx + 2\pi\int_0^1 x\sqrt{1+1^2}\,dx \\
&= 2\pi\sqrt{2}\left(\left[-\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_0^1\right) \\
&= 2\pi\sqrt{2}\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi\sqrt{2}
\end{aligned}$$

2. (20 points)

Consider the parametric curve $x = \sin(\pi t)$, $y = t^2 + 2t$, shown below for $-2.4 \le t \le .4$.



(a) At what point(s) is the tangent line horizontal? *Do not use the picture to justify your answer!*

(b) The curve passes through the origin twice. What are the slopes of the two tangent lines to the curve at the origin?

(c) Find the equation of the form $y = mx + b$ for the tangent at $t = 1$. (This point is not shown in the picture above.)

Solution: (a) We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+2}{\pi\cos(\pi t)},$$

and the tangent line is horizontal when this derivative is 0, namely when $t = -1$. The corresponding Cartesian point is $(\sin(\pi(-1)), (-1)^2 + 2(-1)) = (0, -1)$.

(b) The curve passes through the origin at $t = 0$ and $t = -2$. The two slopes are $2/\pi$ and $-2/\pi$, respectively.

(c) At $t = 1$ we have $x = 0$, $y = 3$, and $dy/dx = -4/\pi$, so the equation for the tangent line is

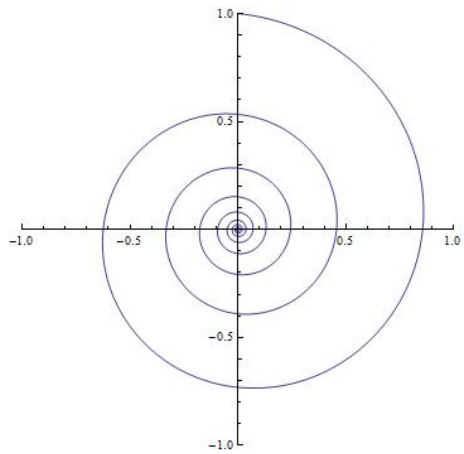
$$y - 3 = -\frac{4}{\pi}(x - 0)$$

or

$$y = -\frac{4}{\pi}x + 3.$$

3. (20 points)

Find the arc length of the parametric curve $x = e^{-t}\sin(10t)\,,\ y = e^{-t}\cos(10t)\,,\ 0 \le t < \infty$, which is the spiral shown below.



Solution: We have

$$dx/dt = e^{-t}(-\sin(10t) + 10\cos(10t)) \qquad \text{and} \qquad dy/dt = e^{-t}(-\cos(10t) - 10\sin(10t)).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= e^{-2t}(-\sin(10t) + 10\cos(10t))^2 + e^{-2t}(-\cos(10t) - 10\sin(10t))^2 \\
&= e^{-2t}(\sin^2(10t) - 20\sin(10t)\cos(10t) + 100\cos^2(10t) \\
&\qquad + \cos^2(10t) + 20\cos(10t)\sin(10t) + 100\sin^2(10t)) \\
&= e^{-2t}(101\sin^2(10t) + 101\cos^2(10t)) \\
&= 101e^{-2t},
\end{aligned}$$

so

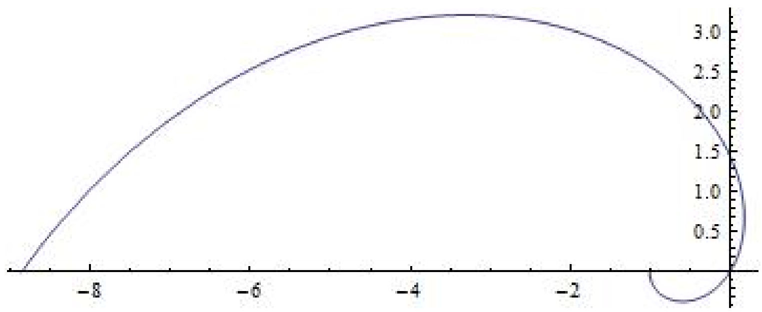
$$\frac{ds}{dt} = \sqrt{101}e^{-t}.$$

By the arc length formula, we have

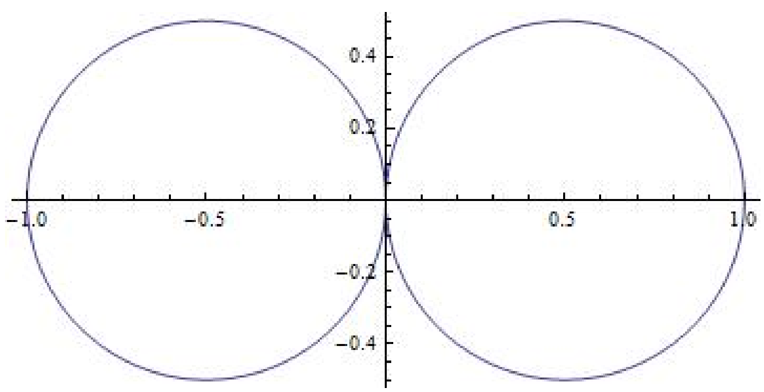
$$\begin{aligned}
L &= \int_0^\infty ds \\
&= \int_0^\infty \sqrt{101}e^{-t}dt \\
&= -\sqrt{101}e^{-t}\Big|_0^\infty \\
&= \sqrt{101}.
\end{aligned}$$

4. (20 points)

(a) Calculate the arc length of the curve $r = \theta^2 - 1$ for $0 \le \theta \le \pi$, which is shown below.



(b) *Use the polar coordinates area formula* to calculate the area enclosed by the curve $r = |\cos\theta|$ for $0 \le \theta \le 2\pi$, which is the "figure eight" shown below.



Solution: (a)

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2}d\theta \\ &= \int_0^{\pi} \sqrt{(\theta^2 - 1)^2 + (2\theta)^2}d\theta \\ &= \int_0^{\pi} \sqrt{(\theta^2 + 1)^2}d\theta \\ &= \int_0^{\pi} (\theta^2 + 1)d\theta \\ &= \left[\frac{\theta^3}{3} + \theta\right]_0^{\pi} \\ &= \frac{\pi^3}{3} + \pi \end{aligned}$$

Solution: (b) We need to treat the cases where $\cos\theta$ is positive and negative seaprately. They correspond to the right and left hand circles. It suffices to find the area of the left hand circle, for which $\pi/2 \le \theta \le 3\pi/2$, and double it. Thus we have

$$\begin{aligned} A &= 2\int_{\pi/2}^{3\pi/2} \frac{r^2}{2} d\theta \\ &= \int_{\pi/2}^{3\pi/2} \cos^2\theta d\theta \\ &= \int_{\pi/2}^{3\pi/2} \frac{1+\cos 2\theta}{2} d\theta \\ &= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{3\pi/2} \\ &= \frac{\pi}{2}. \end{aligned}$$

5. (20 points)

(a) (5 points) Does the sequence $\{a_n : n \ge 1\}$ with $a_n = 1/\sqrt[3]{n}$ converge? Why or why not?

(b) (5 points) Use L'Hôpital's Rule to show that for $k > 0$,

$$\lim_{x\to\infty} x^k \, e^{-x} = k \lim_{x\to\infty} x^{k-1} \, e^{-x} \, .$$

(c) (5 points) Let $b_n = n^3 \, e^{-n}$. Show that the sequence $\{b_n : n \ge 1\}$ converges. What is the limit?

(d) (5 points) Does the sequence $\{c_n : n \ge 1\}$ with

$$c_n = \left(-\frac{1}{2}\right)^n \cos\left(\frac{n\pi}{3}\right)$$

converge? Why or why not?

Solution:

(a) Since $\lim_{n\to\infty} \sqrt[3]{n} = \infty$, $\lim_{n\to\infty} a_n = 0$ and the sequence converges to 0.

(b) We have

$$\begin{aligned}\lim_{x\to\infty} x^k e^{-x} &= \lim_{x\to\infty} \frac{x^k}{e^x} \\ &= \lim_{x\to\infty} \frac{kx^{k-1}}{e^x} \qquad \text{by L'Hôpital's Rule} \\ &= k \lim_{x\to\infty} \frac{x^{k-1}}{e^x} \\ &= k \lim_{x\to\infty} x^{k-1}e^{-x}.\end{aligned}$$

(c) From (b) we see that

$$\lim_{x\to\infty} x^3 e^{-x} = 3 \lim_{x\to\infty} x^2 e^{-x} = 6 \lim_{x\to\infty} x e^{-x} = 6 \lim_{x\to\infty} e^{-x} = 0,$$

so the sequence converges to 0.

(d) Since $-1 \le \cos(n\pi/3) \le 1$, we have $-1/2^n \le c_n \le 1/2^n$, so

$$0 = \lim_{n\to\infty} \frac{-1}{2^n} \le \lim_{n\to\infty} c_n \le \lim_{n\to\infty} \frac{1}{2^n} = 0$$

and $\lim_{n\to\infty} c_n = 0$.