Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 16, 2009

1. (20 points)

What is the surface area of the surface of revolution obtained by rotating the infinite curve e^{-x}, $x \geq 0$ around the x-axis?

You may use the formula

$$\int \sec^3(x)\, dx = \frac{\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|}{2} + C.$$

Solution:

We have $y' = -e^{-x}$ so

$$\begin{aligned}
A &= 2\pi \int_0^\infty y\sqrt{1+y'^2}dx \\
&= 2\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}dx \\
&= -2\pi \int_1^0 \sqrt{1+u^2}du && \text{where } u = e^{-x} \\
& && du = -e^{-x}dx \\
&= 2\pi \int_0^1 \sqrt{1+u^2}du \\
&= 2\pi \int_0^{\pi/4} \sec^3\theta d\theta && \text{where } u = \tan\theta \\
& && \text{so } du = \sec^2\theta d\theta \\
& && \text{and } \sqrt{1+u^2} = \sec\theta \\
&= \pi\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4} \\
&= \pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right).
\end{aligned}$$

2. (20 points)

Consider the parametric curve (an astroid or 4 pointed hypocycloid) $x = \cos^3(t)$, $y = \sin^3(t)$, $t \in [0, 2\pi]$.

(a) (7 points) At what points is the tangent horizontal or vertical?

(b) (6 points) At what points does it have slope ± 1?

(c) (7 points) Find the equation of the form $y = mx + b$ for the tangent at $t = \frac{\pi}{4}$.

Solution: (a) We have

$$\begin{aligned} \frac{dx}{dt} &= -3 \sin t \cos^2 t \\ \frac{dy}{dt} &= 3 \cos t \sin^2 t \\ \frac{dy}{dx} &= -\frac{\cos t \sin^2 t}{\sin t \cos^2 t} = -\tan t \end{aligned}$$

The tangent line is horizontal when this derivative is 0, namely when $t = 0$ and $t = \pi$. The tangent line is vertical when the derivative is undefined, namely at $t = \pi/2$ and $t = 3\pi/2$.

Solution: (b) The slope of the tangent line is ± 1 when $t = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$.

Solution: (c) At $t = \pi/4$ we have $x = y = \sqrt{2}/4$ and $dy/dx = -1$, so the equation for the tangent line is

$$\begin{aligned} \frac{y - \sqrt{2}/4}{x - \sqrt{2}/4} &= -1 \\ y - \sqrt{2}/4 &= -(x - \sqrt{2}/4) \\ &= -x + \sqrt{2}/4 \\ y &= -x + \sqrt{2}/2. \end{aligned}$$

3. (20 points)

Find the arc length of the cycloid $x = r(t - \sin(t))$ and $y = r(1 - \cos(t))$, for $0 \leq t \leq 2\pi$.

Solution: We have

$$\begin{aligned}
\frac{dx}{dt} &= r(1 - \cos t) \\
\frac{dy}{dt} &= r \sin t) \\
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
&= r^2 \left((1 - \cos t)^2 + \sin^2 t\right) \\
&= r^2 \left(1 - 2\cos t + \cos^2 t + \sin^2 t\right) \\
&= 2r^2 (1 - \cos t) \\
\frac{ds}{dt} &= 2r\sqrt{\frac{1 - \cos t}{2}} \\
&= 2r \sin(t/2),
\end{aligned}$$

so the arc length is

$$\begin{aligned}
s &= \int_0^{2\pi} 2r \sin(t/2) dt \\
&= 4r \int_0^{\pi} \sin u du \qquad \text{where } u = t/2 \\
&= -4r \cos u|_0^{\pi} \\
&= 8r
\end{aligned}$$

4. (20 points)

Consider the logarithmic spiral $r = e^{\theta}$, $\theta \geq 0$, which can be defined parametrically by $x = e^t \cos t$ and $y = e^t \sin t$ with $t = \theta$.

(a) (10 points) Calculate the arc-length of the logarithmic spiral for $0 \leq \theta \leq b$.

(b) (10 points) Calculate the area of the region between the x-axis and the curve for $0 \leq \theta \leq \pi$.

Solution: (a) For the arc length we have

$$\begin{aligned}
\frac{dx}{dt} &= e^t(\cos t - \sin t) \\
\frac{dy}{dt} &= e^t(\sin t + \cos t) \\
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
&= e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right) \\
&= e^{2t}\left((\cos^2 - 2\cos t \sin t + \sin^2 t) + (\cos^2 + 2\cos t \sin t + \sin^2 t)\right) \\
&= 2e^{2t} \\
\frac{ds}{dt} &= e^t\sqrt{2},
\end{aligned}$$

so

$$\begin{aligned}
s &= \sqrt{2}\int_0^b e^t dt \\
&= \sqrt{2}\, e^t\Big|_0^b \\
&= \sqrt{2}(e^b - 1).
\end{aligned}$$

Solution: (b) Using the area formula for polar coordinates, we have

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} r^2 d\theta \\
&= \frac{1}{2}\int_0^{\pi} e^{2\theta} d\theta \\
&= \frac{1}{2}\,\frac{e^{2\theta}}{2}\Big|_0^{\pi} \\
&= \frac{e^{2\pi} - 1}{4}
\end{aligned}$$

5. (20 points)

(a) (5 points) Use L'Hospital's Rule to show that for $k > 0$,

$$\lim_{x\to\infty} x^k \, e^{-x^2} = \frac{k}{2} \lim_{x\to\infty} x^{k-2} \, e^{-x^2} \, .$$

(b) (5 points) Let $a_n = n^8 \, e^{-n^2}$ where $n = 1, 2, 3, \ldots$. Show that the sequence $\{a_n : n \geq 1\}$ converges. What is the limit?

(c) (5 points) Does the sequence $b_n = \cos(\frac{n\pi}{2})(-\frac{1}{2})^n$ converge? Why or why not?

(d) (5 points) Does the sequence $b_n = \frac{1}{n^{0.005}}$ converge? Why or why not?

Solution: (a) We have

$$\begin{aligned}\lim_{x\to\infty} x^k \, e^{-x^2} &= \lim_{x\to\infty} \frac{x^k}{e^{x^2}} \\ &= \lim_{x\to\infty} \frac{kx^{k-1}}{2x \, e^{x^2}} \\ &= \frac{k}{2} \lim_{x\to\infty} \frac{x^{k-2}}{e^{x^2}} \\ &= \frac{k}{2} \lim_{x\to\infty} x^{k-2} \, e^{-x^2},\end{aligned}$$

Solution: (b) From (a) we see that

$$\lim_{x\to\infty} x^8 \, e^{-x^2} = 4 \lim_{x\to\infty} x^6 \, e^{-x^2} = 12 \lim_{x\to\infty} x^4 \, e^{-x^2} = 24 \lim_{x\to\infty} x^2 \, e^{-x^2} = 24 \lim_{x\to\infty} e^{-x^2} = 0,$$

so the sequence converges to 0.

Solution: (c) Since $-1 \leq \cos(n\pi/2) \leq 1$, $-1/2^n \leq b_n \leq 1/2^n$, so $\lim_{n\to\infty} b_n = 0$.

Solution: (d) Since $\lim_{n\to\infty} n^{.005} = \infty$, $\lim_{n\to\infty} b_n = 0$.