Math 162: Calculus IIA

Second Midterm Exam Solutions

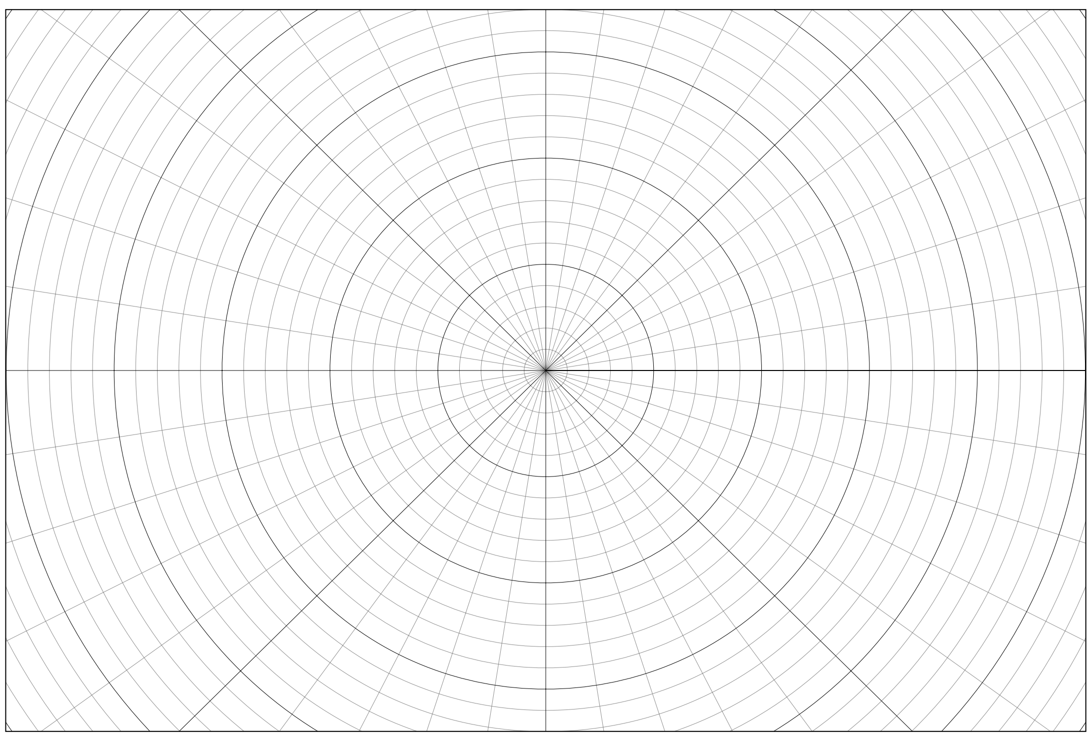
November 20, 2008

Part A

1. (15 points) Consider the polar function

$$r(\theta) = 1 + \sin(2\theta)$$

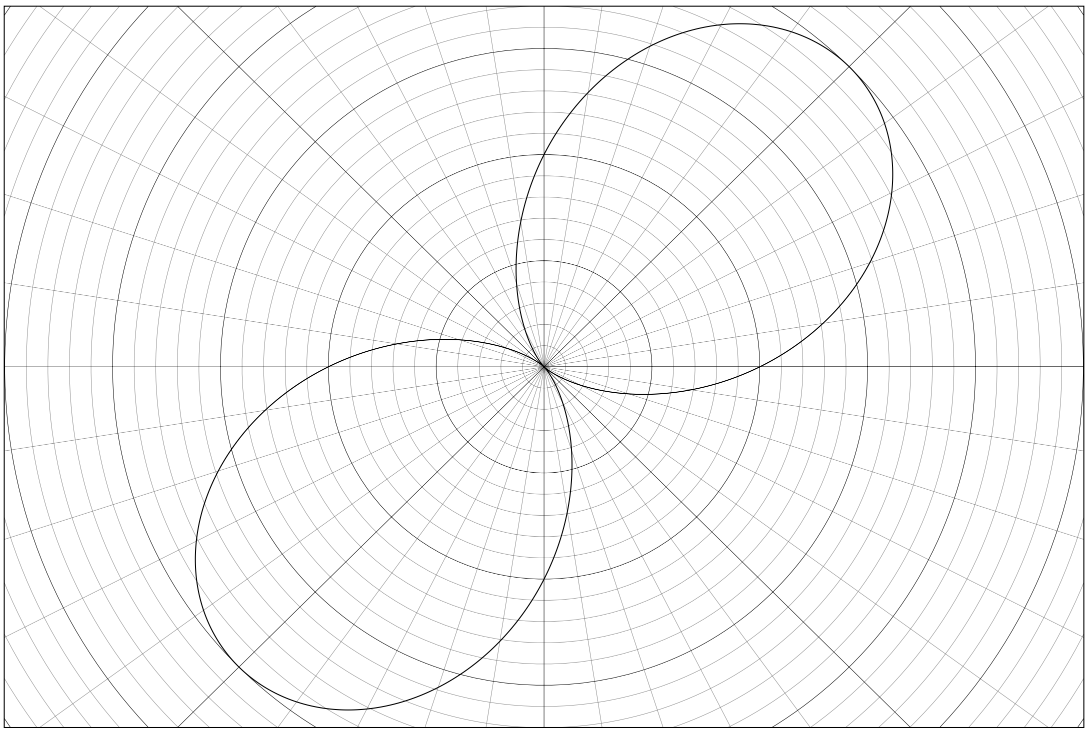
(a) Sketch the graph of the "two-paddled fan" on the provided axes.



(b) Set up (but do not evaluate) the integral representing the area of one paddle of the fan.

(c) Set up (but do not evaluate) the integral for the perimeter of one paddle of the fan.

Solution:



(a)

(b) *Determining the bounds.* The bounds will be between angles in radian measurement given $r = 0$. To find these angles then, we must solve the equation:

$$\begin{aligned} 0 &= 1 + \sin(2\theta) \\ -1 &= \sin(2\theta) \\ \Rightarrow 2\theta &= \frac{3\pi}{2} + 2\pi k \\ \theta &= \frac{3\pi}{4} + \pi k \end{aligned}$$

Therefore, the bounds we will take are $\theta_1 = \frac{3\pi}{4}$ to $\theta_2 = \frac{7\pi}{4}$. Alternatively, for example, you could have used $-\frac{\pi}{4}$ to $\frac{3\pi}{4}$.

Setting up the integral. Recall that for a polar function, the area is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

Substituting into this formula, we get:

$$\boxed{A = \int_{3\pi/4}^{7\pi/4} \frac{1}{2}\left(1+\sin 2\theta\right)^2\, d\theta}$$

(c) Recall the arc length formula for polar curves is given by

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The derivative $r' = 2\cos(2\theta)$, and the bounds are as in part (b). This gives the integral

$$\boxed{s = \int_{3\pi/4}^{7\pi/4} \sqrt{(1+\sin(2\theta))^2 + 4\cos^2(2\theta)}\, d\theta}$$

2. (25 points) Consider the graph of the cycloid given parametrically by

$$\begin{cases} x(t) &= 2(t-\sin t) \\ y(t) &= 2(1-\cos t) \end{cases}$$

(a) What is the area under one arch of the cycloid?

(b) What is the length of one arch of the cycloid?
Hint: $1-\cos t = 2\sin^2(\frac{t}{2})$.

(c) Find the equation of the line tangent to the cycloid at $t = \frac{\pi}{3}$.

Solution:

(a) Recall the area under a graph of y is given by

$$A = \int_a^b y\, dx$$

but in terms of parametric curves, x and y are functions of the variable t, where t is varying from 0 to 2π (one complete revolution). This formula becomes:

$$\begin{aligned}
A &= \int_\alpha^\beta y(t) \cdot x'(t)\, dt \\
&= \int_0^{2\pi} (2(1-\cos t)) \cdot (2(1-\cos t))\, dt \\
&= \int_0^{2\pi} 4(1-\cos t)^2\, dt \\
&= \int_0^{2\pi} 4(1 - 2\cos t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} 4\left(1 - 2\cos t + \frac{1}{2}(1+\cos(2t))\right)\, dt \\
&= \int_0^{2\pi} 4\left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos(2t)\right)\, dt \\
&= 4\left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin(2t)\right)\Big|_0^{2\pi} \\
&= 4\left(\frac{3}{2}(2\pi) - 0 + \frac{1}{4}(0)\right) - 4(0-0+0) \\
&= \boxed{12\pi}
\end{aligned}$$

(b) The arclength formula for a parametric equation is given by

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Therefore, the integral proceeds as follows:

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{(2(1-\cos t))^2 + (2\sin t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{4(1 - 2\cos t + \cos^2 t) + 4\sin^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{4 - 8\cos t + (4\cos^2 t + 4\sin^2 t)}\, dt \\
&= \int_0^{2\pi} \sqrt{8 - 8\cos t}\, dt \\
&= \int_0^{2\pi} \sqrt{8}\sqrt{1-\cos t}\, dt \\
&= \int_0^{2\pi} \sqrt{8}\sqrt{2\sin^2\left(\frac{t}{2}\right)}\, dt \\
&= \int_0^{2\pi} 4\sin\left(\frac{t}{2}\right)\, dt \\
&= -8\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} \\
&= -8(\cos(\pi) - \cos(0)) \\
&= -8(-1-1) \\
&= \boxed{16}
\end{aligned}$$

(c) Recall that the derivative of a parametric function is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Hence the derivative is:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2\sin t}{2(1-\cos t)} \\
&= \frac{\sin t}{1-\cos t}
\end{aligned}$$

So that the slope at $t = \frac{\pi}{3}$ is

$$\begin{aligned}
\frac{dy}{dx}\left(\frac{\pi}{3}\right) &= \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} \\
&= \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} \\
&= \frac{\sqrt{3}}{2-1} \\
&= \sqrt{3}
\end{aligned}$$

The point corresponding to $t = \frac{\pi}{3}$ is

$$x = 2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = \frac{2\pi}{3} - 2\cdot\frac{\sqrt{3}}{2} = \frac{2\pi}{3} - \sqrt{3}$$
$$y = 2\left(1 - \cos\frac{\pi}{3}\right) = 2 - 2\cdot\frac{1}{2} = 1$$

Therefore, the equation is:

$$\boxed{y - 1 = \sqrt{3}\left(x - \frac{2\pi}{3} + \sqrt{3}\right)}$$

3. (15 points) Consider the sequence whose n-th term is $a_n = ne^{-n}$.

(a) Determine if the sequence is increasing, decreasing, or not monotonic.

(b) Is the sequence bounded?

(c) Is the sequence convergent? If it is, what is the limit?

Solution: The function $f(x) = xe^{-x}$ is positive and decreasing (look at the derivative!) on $[1, \infty)$, hence the sequence is decreasing. Note then that the sequence is bounded by the 1-st term in the sequence: $\frac{1}{e}$. A monotonic bounded sequence is convergent. Moreover, using L'Hospital rule:

$$\begin{aligned} \lim_{x\to\infty} xe^{-x} &= \lim_{x\to\infty} \frac{x}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{e^x} \\ &= \boxed{0} \end{aligned}$$

Therefore $a_n \to 0$.

4. (15 points) Determine if the following series are convergent, and if they are, find their sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Solution:

(a) The series

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1}$$

is a geometric series with $a = e$ and $r = \frac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is:

$$\frac{e}{1-\frac{e}{3}} = \boxed{\frac{3e}{3-e}}$$

(b) The series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$$

is a telescoping series and the n-th partial sum

$$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln(n-1) - \ln(n)) + (\ln(n) - \ln(n+1))$$

Therefore

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln 1 - \ln(n+1) = \lim_{n\to\infty} -\ln(n+1) = -\infty$$

Therefore the series diverges.

5. (15 points)

(a) Does the following series converge? Why or why not?

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

(b) If it does, how big is the error when using s_5, the 5-th partial sum of the series, as an approximation to the sum.

Solution:

(a) Let $f(x) = x^2 e^{-x^3}$. Then f is continous, positive and decreasing (look at the deivative!) on $[1, \infty)$ and we can apply the integral test.

$$\int_1^{\infty} x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$$

Therefore the integral converges and so does the series.

(b) The error in using s_5, the 5-th partial sum of the series, as an approximation to the sum is R_5. When using the integral test

$$R_5 \leq \int_5^\infty x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \left[-\frac{1}{3} e^{-x^3} \right]_5^t = \boxed{\frac{1}{3e^{125}}}$$

6. (15 points) Do the following series converge? Why or why not?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

Solution:

(a) Let $b_n = \frac{n}{n^2+2}$. Then, the sequence $\{b_n\}$ is decreasing for $n \geq 2$, since $\left(\frac{x}{x+2}\right)' < 0$ for $x \geq \sqrt{2}$. Also,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{2}{n^2}} = 0$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ converges by the alternating series test.

(b) Using the limit comparison test with $a_n = \frac{n^2+1}{n^3+1}$ and $b_n = \frac{1}{n}$, we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+1}{n^3+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3+n}{n^3+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}} = 1 > 0$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges (it is the harmonic series!), then the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ is also divergent.

(c) If $a_k = \frac{5^k}{3^k+4^k}$, then

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5^k}{3^k+4^k} = \lim_{k\to\infty} \frac{\left(\frac{5}{4}\right)^k}{\left(\frac{3}{4}\right)^k + 1} = \infty$$

Thus, the series $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$ diverges by the divergence test.