MATH 162

Midterm 2 ANSWERS

November 16, 2006

1. (11 points)

Express the following continued fraction as a rational number.

$$82.626262\ldots$$

If you can express it as the sum of two rational numbers, that's also OK.

Answer:

Espressing the number in terms of a geometric series, we find

$$\begin{aligned}
82.626262\ldots &= 80 + \frac{26.262\ldots}{10} \\
&= 80 + \frac{26}{10}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n \\
&= 80 + \frac{26}{10}\cdot\frac{1}{1-(1/100)} \\
&= 80 + \frac{26}{10}\cdot\frac{1}{99/100} \\
&= 80 + \frac{26}{10}\cdot\frac{100}{99} \\
&= 80 + \frac{260}{99}
\end{aligned}$$

You could also do it this way:

$$\begin{aligned}
82.626262\ldots &= 82 + \frac{62.62\ldots}{100} \\
&= 82 + \frac{62}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n \\
&= 82 + \frac{62}{100}\cdot\frac{1}{1-(1/100)} \\
&= 82 + \frac{62}{100}\cdot\frac{1}{99/100} \\
&= 82 + \frac{62}{100}\cdot\frac{100}{99} \\
&= 82 + \frac{62}{99}
\end{aligned}$$

2. (11 points)

Find the sum of the following series.

$$\sum_{n=1}^{\infty}\frac{2}{n^2+3n}$$

Hint: Use partial fractions.

Answer:

Using partial fractions, we get

$$\frac{2}{n^2+3n} = \frac{2}{3}\cdot\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{2}{n^2+3n} = \frac{2}{3}\cdot\left[\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\ldots\right]$$

This is a telescoping series, and all but three terms cancel out, so

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{2}{n^2+3n} &= \frac{2}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right) \\
&= \frac{2}{3}\cdot\frac{11}{6} \\
&= \frac{11}{9}
\end{aligned}$$

3. (11 points)

(a) (5 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^2+3}}$$

Answer:

The series diverges

(b) (6 points) Why or why not?

Answer:

If we look at the dominant terms, we get $2n/\sqrt{n^2} = 2$, so we suspect that the series diverges by the divergence test, since the limit of the terms is not 0. To make this rigorous, we divide top and bottom by the leading power of n, namely just n, to get

$$\frac{2n}{\sqrt{n^2+3}} = \frac{1/n}{1/n} \cdot \frac{2n}{\sqrt{n^2+3}} = \frac{2}{\sqrt{1+3/n^2}} \to 2$$

as $n \to \infty$. So, the series diverges by the divergence test.

4. (11 points)

(a) (5 points) Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Answer:

The series converges.

(b) (6 points) Justify your answer in part (a).

Answer:

This series can be analyzed using the integral test. Using the substitution $u = \ln x$, $du =$

$(1/x)dx$ we get

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^2}\,dx &= \int_{\ln 2}^\infty \frac{1}{u^2}\,du \\ &= -\frac{1}{u}\Big|_{\ln 2}^\infty \\ &= 0 - \left(-\frac{1}{\ln 2}\right) \\ &< \infty\end{aligned}$$

and so the series converges.

5. (12 points)

Consider the series

$$\sum_{n=1}^\infty (-1)^n \frac{1}{n^2}$$

(a) (3 points) Does this series converge or diverge?

Answer:

The series converges.

(b) (4 points) Justify your answer in part (a).

Answer:

The series is an alternating series with terms which decrease to 0. Therefore, it converges by the alternating series test.

(c) (5 points) Suppose we approximate the series by taking the sum of the first n terms, up to and including $(-1)^n(1/n^2)$. What is the first value of n for which our error is less than or equal to $1/10^4$?

Answer:

For a convergent alternating series, we can estimate the remainder as follows. Recall that S is the sum of the series, and S_n is the sum up to and including the nth term. Then

$$|S - S_n| \leq b_{n+1}$$

In our case, $b_{n+1} = 1/(n+1)^2$. But the first value of n for which $1/(n+1)^2 \leq 1/10^4$ is when $n + 1 = 100$, or

$$n = 99$$

6. (11 points)

(a) (5 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n + 7}{2^n - 1}$$

Answer:

The series diverges.

(b) (6 points) Justify your answer in part (a), making sure to name any convergence tests that you are using.

Answer:

We can use the Comparison Test and compare this series with $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$. Note that

$$\frac{3^n + 7}{2^n - 1} > \frac{3^n}{2^n}.$$

The series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a geometric series with $\frac{3}{2} > 1$, hence it is divergent. Then by the Comparison Test, $\sum_{n=1}^{\infty} \frac{3^n + 7}{2^n - 1}$ is also divergent.

One can also use the Limit Comparison Test to compare the given series with $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{3^n+7}{2^n-1}}{\frac{3^n}{2^n}} &= \lim_{n\to\infty} \frac{3^n + 7}{2^n - 1} \cdot \frac{2^n}{3^n} \\
&= \lim_{n\to\infty} \frac{3^n + 7}{3^n} \cdot \frac{2^n}{2^n - 1} \\
&= \lim_{n\to\infty} \left(1 + \frac{7}{3^n}\right) \left(\frac{1}{1 - \frac{1}{2^n}}\right) \\
&= 1
\end{aligned}$$

Therefore, since the limit is a number greater than zero and $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges, the given series must also diverge.

7. (11 points)

(a) (5 points) Does this series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\pi^{n+3}}{n^{n/2}}$$

Answer:

The series converges.

(b) (6 points) Justify your answer in part (a), making sure to name any convergence tests that you are using.

Answer:

First we re-write the series as:

$$\sum_{n=1}^{\infty} \frac{\pi^{n+3}}{n^{n/2}} = \pi^3 \sum_{n=1}^{\infty} \left(\frac{\pi}{\sqrt{n}} \right)^n$$

We now use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{\pi}{\sqrt{n}} \right)^n} = \lim_{n\to\infty} \frac{\pi}{\sqrt{n}} = 0$$

Since the limit is less than 1, by the Root Test, the series $\sum_{n=1}^{\infty} \left(\frac{\pi}{\sqrt{n}} \right)^n$ converges, and so does the given series.

8. (11 points)

(a) (5 points) Does this series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n \sqrt{n+2}}{7(2n)!}$$

Answer:

The series converges.

(b) (6 points) Justify your answer, making sure to name any convergence tests that you are using.

Answer:

Since the terms of the series are made up of factorials and exponentials, it makes sense we use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}\sqrt{n+3}}{7(2n+2)!}}{\frac{3^n\sqrt{n+2}}{7(2n)!}} = \lim_{n\to\infty} \frac{3^{n+1}\sqrt{n+3}}{7(2n+2)!} \cdot \frac{7(2n)!}{3^n\sqrt{n+2}}$$
$$= \lim_{n\to\infty} 3\frac{\sqrt{n+3}}{\sqrt{n+2}} \cdot \frac{(2n)!}{(2n+2)!}$$
$$= \lim_{n\to\infty} 3\sqrt{\frac{n+3}{n+2}} \cdot \frac{1}{(2n+2)(2n+1)}$$
$$= 3 \cdot 1 \cdot 0$$

This limit is less than 1, and so the series converges by the Ratio Test.

9. (11 points)

Find the limit of this sequence.

$$\lim_{n\to\infty} n \tan\left(\frac{1}{n}\right)$$

Answer:

First, we make the change of variable $x = 1/n$, so that $n = 1/x$. As $n \to \infty$, we have $x \to 0$. Then

$$\lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = \lim_{x\to 0} \frac{\tan x}{x}$$

From here we could use L'Hopital's rule. Since $(\tan x)' = \sec^2 x$ and $\sec 0 = 1/\cos 0 = 1$, and since $\sec x$ is continuous at $x = 0$, we have

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$$

We could also use the fact that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \cos x = 1$$

to argue that

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1$$