

Math 162: Calculus IIA

Midterm Exam 2 ANSWERS

April 6, 2007

Part A

1. (11 points)

Compute the following integral

$$\int \frac{x+1}{\sqrt{4-x^2}} dx$$

We use trig substitution with

$$x = 2 \sin \theta$$

$$dx = 2 \cos \theta d\theta$$

$$4 - x^2 = 4 - 4 \sin^2 \theta = 4 \cos^2 \theta$$

Then the integral we have to solve is

$$\begin{aligned} \int \frac{x+1}{\sqrt{4-x^2}} dx &= \int \frac{(2 \sin \theta + 1)2 \cos \theta d\theta}{2 \cos \theta} \\ &= \int (2 \sin \theta + 1) d\theta \\ &= -2 \cos \theta + \theta + C \end{aligned}$$

Now we have to plug back in for θ . From the equations above, or using the triangle, we have $\cos \theta = \sqrt{4-x^2}$, and of course $\theta = \arcsin\left(\frac{x}{2}\right)$. Therefore,

$$\int \frac{x+1}{\sqrt{4-x^2}} dx = -2\sqrt{4-x^2} + \arcsin\left(\frac{x}{2}\right) + C.$$

2. (11 points)

Compute the following integral

$$\int \frac{4}{x^2 + 4x + 3} dx$$

Note that $x^2 + 4x + 3 = (x+1)(x+3)$, so using partial fractions, we get

$$\frac{4}{x^2 + 4x + 3} = \frac{A}{x+1} + \frac{B}{x+3}$$

Solving for A and B in the equation

$$4 = A(x + 3) + B(x + 1)$$

we get $A = 2$, $B = -2$. Therefore,

$$\begin{aligned} \int \frac{4}{x^2 + 4x + 3} dx &= \int \left(\frac{2}{x+1} - \frac{2}{x+3} \right) dx \\ &= 2 \int \frac{dx}{x+1} - 2 \int \frac{dx}{x+3} \\ &= 2 \ln|x+1| - 2 \ln|x+3| + C \end{aligned}$$

3. (11 points)

Find the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + 4n + 3}$$

Hint: Use partial fractions.

Using partial fractions as in the previous problem, we get

$$\frac{4}{n^2 + 4n + 3} = \frac{2}{n+1} - \frac{2}{n+3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + 4n + 3} = \left(\frac{2}{2} - \frac{2}{4} \right) + \left(\frac{2}{3} - \frac{2}{5} \right) + \left(\frac{2}{4} - \frac{2}{6} \right) + \left(\frac{2}{5} - \frac{2}{7} \right) + \dots$$

This is a telescoping series, and all but two terms cancel out, so

$$\sum_{n=1}^{\infty} \frac{4}{n^2 + 4n + 3} = \frac{2}{2} + \frac{2}{3} = \frac{5}{3}$$

4. (11 points)

Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \cos \left(n + \frac{1}{n} \right)$$

Justify your answer, making sure to name the convergence test(s) that you are using.

The series diverges by the Divergence Test:

$$\lim_{n \rightarrow \infty} \cos \left(n + \frac{1}{n} \right) = dne.$$

Since $\lim_{n \rightarrow \infty} \cos \left(n + \frac{1}{n} \right) \neq 0$, the series diverges.

5. (11 points)

Does the following series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^7}$$

Justify your answer, making sure to name the convergence test(s) that you are using.

This series can be analyzed using the integral test. Using the substitution $u = \ln x$, $du = (1/x)dx$ we get

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^7} dx &= \int_{\ln 2}^{\infty} \frac{1}{u^7} du \\ &= \lim_{t \rightarrow \infty} \int_{\ln 2}^t \frac{1}{u^7} du \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{6u^6} \Big|_{\ln 2}^t \right) \\ &= \lim_{t \rightarrow \infty} \left(\frac{1}{6(\ln 2)^6} - \frac{1}{6t^6} \right) \\ &= \frac{1}{6(\ln 2)^6} - 0 \\ &< \infty \end{aligned}$$

and so the series converges.

6. (12 points)

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$$

(a) (5 points) Does this series converge or diverge? Justify your answer, making sure to name any convergence tests that you are using.

The series is an alternating series with terms which decrease to 0. Therefore, it converges by the alternating series test.

(b) (7 points) Suppose we approximate the series by taking the sum of the first n terms, up to and including $(-1)^n(1/n^3)$. What is the first value of n for which our error is less than or equal to $1/10^6$?

For a convergent alternating series, we can estimate the remainder as follows. Recall that S is the sum of the series, and S_n is the sum up to and including the n th term. Then

$$|S - S_n| \leq b_{n+1}$$

In our case, $b_{n+1} = 1/(n+1)^3$. But the first value of n for which $1/(n+1)^3 \leq 1/10^6$ is when $n+1 = 100$, or

$$n = 99.$$

7. (11 points)

Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{5^n + 8}{2^n - 1}$$

Justify your answer, making sure to name any convergence tests that you are using.

We can use the Comparison Test and compare this series with $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$. Note that

$$\frac{5^n + 8}{2^n - 1} > \frac{5^n}{2^n}.$$

The series $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ is a geometric series with $\frac{5}{2} > 1$, hence it is divergent. Then by the Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n + 8}{2^n - 1}$ is also divergent.

One can also use the Limit Comparison Test to compare the given series with $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{5^n + 8}{2^n - 1}}{\frac{5^n}{2^n}} &= \lim_{n \rightarrow \infty} \frac{5^n + 8}{2^n - 1} \cdot \frac{2^n}{5^n} \\ &= \lim_{n \rightarrow \infty} \frac{5^n + 8}{5^n} \cdot \frac{2^n}{2^n - 1} \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{8}{5^n}\right) \left(\frac{1}{1 - \frac{1}{2^n}}\right) \\ &= 1 \end{aligned}$$

Therefore, since the limit is a number greater than zero and $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ diverges, the given series must also diverge.

8. (11 points)

Does this series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{5^n \sqrt{n+5}}{3(2n)!}$$

Justify your answer, making sure to name any convergence tests that you are using.

Since the terms of the series are made up of factorials and exponentials, it makes sense we use the Ratio Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{5^{n+1} \sqrt{n+6}}{3(2n+2)!}}{\frac{5^n \sqrt{n+5}}{3(2n)!}} &= \lim_{n \rightarrow \infty} \frac{5^{n+1} \sqrt{n+6}}{3(2n+2)!} \cdot \frac{3(2n)!}{5^n \sqrt{n+5}} \\ &= \lim_{n \rightarrow \infty} 5 \frac{\sqrt{n+6}}{\sqrt{n+5}} \cdot \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n \rightarrow \infty} 5 \sqrt{\frac{n+6}{n+5}} \cdot \frac{1}{(2n+2)(2n+1)} \\ &= 5 \cdot 1 \cdot 0 \end{aligned}$$

This limit is less than 1, and so the series converges by the Ratio Test.

9. (11 points)

Find the limit of this sequence.

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{\sqrt{n^4 + 1}}$$

We will divide the numerator and denominator each by n^2 . This means dividing by n^4 inside the square root sign. Thus we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^2 + 1}{\sqrt{n^4 + 1}} &= \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{\sqrt{1 + 1/n^4}} \\ &= \frac{\lim_{n \rightarrow \infty} (1 + 1/n^2)}{\lim_{n \rightarrow \infty} \sqrt{(1 + 1/n^4)}} \\ &= 1. \end{aligned}$$