Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 16, 2007

1. (15 points) Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{2}{3^n}$$

Expressing the series in terms of a geometric series, we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2}{3^n} &= 2 + \sum_{n=1}^{\infty} \frac{2}{3^n} \\ &= 2 + \sum_{n=1}^{\infty} \frac{2}{3} \cdot \left(\frac{1}{3}\right)^{n-1}\end{aligned}$$

Thus the series becomes a number plus a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3} < 1$, hence it converges and:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2}{3^n} &= 2 + \frac{\frac{2}{3}}{1 - \frac{1}{3}} \\ &= 2 + 1 \\ &= 3\end{aligned}$$

2. (15 points) Find the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2 + 2n}$$

Hint: Use partial fractions.

Using partial fractions, we get

$$\frac{3}{n^2 + 2n} = \frac{3}{2} \cdot \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

Therefore, the n-th partial sum looks like:

$$\sum_{i=1}^{n} \frac{3}{i^2+2i} = \frac{3}{2} \cdot \left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) \right]$$
$$= \frac{3}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

Since in the telescopic sum, all but these four summands cancel out. Then

$$s = \sum_{n=1}^{\infty} \frac{3}{n^2+2n} = \lim_{n\to\infty} s_n = \frac{3}{2}\left(\frac{3}{2}\right) = \frac{9}{4}$$

3. (15 points)

Does the following series converge or diverge? Why?

$$\sum_{n=1}^{\infty} \frac{2n}{\ln(n+1)}$$

Justify your answer, making sure to name any convergence or divergence tests that you are using.

Let's look at the n-th term of the sum: $a_n = \frac{2n}{\ln(n+1)}$ Then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{\ln(n+1)}$$

This is a limit with indefinite form $\frac{\infty}{\infty}$. Thus applying L'Hospital's rule we get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{\frac{1}{n+1}} = \lim_{n\to\infty} 2(n+1) = \infty$$

So, the series diverges by the divergence test.

4. (15 points)

Does the following series converge conditionally, converge absolutely or diverge? Why?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$$

Justify your answer, making sure to name any convergence or divergence tests that you are using. The series converges by the alternating seris test since it is alternating and the terms are decreasing to zero.

This series of absolute values can be analyzed using the integral test. Using the substitution $u = \ln x$, $du = (1/x)dx$ we get

$$\begin{aligned}\int_1^\infty \frac{1}{x \ln x}\, dx &= \int_0^\infty \frac{du}{u} \\ &= \ln u \Big|_0^\infty \\ &= \infty\end{aligned}$$

and so the series diverges. Hence the original series converges conditionally but not absolutely.

5. (15 points) Find the radius and interval of convergence of the following power series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Using the ratio test, where $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$, then

$$\begin{aligned}|\frac{a_{n+1}}{a_n}| &= \frac{x^{2(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{x^{2n}} \\ &= \frac{x^{2n} \cdot x^2}{(2n+2) \cdot (2n+1) \cdot (2n)!} \cdot \frac{(2n)!}{x^{2n}} \\ &= \frac{x^2}{(2n+2) \cdot (2n+1)}\end{aligned}$$

Therefore:

$$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| = \lim_{n \to \infty} \frac{x^2}{(2n+2) \cdot (2n+1)} = 0 < 1$$

Thus the series converges for all values of x. So the radius on convergence is $R = \infty$ and the interval on convergence is $(-\infty, \infty)$.

6. (15 points)

Does this series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{2^n (n+1)}{n!}$$

Justify your answer, making sure to name any convergence or divergence tests that you are using.

Since the terms of the series are made up of factorials and exponentials, it makes sense we use the Ratio Test:

$$\begin{aligned} a_{n+1}a_n &= \frac{2^{n+1}(n+2)}{(n+1)!} \Big/ \frac{2^n(n+1)}{n!} \\ &= \frac{2^{n+1}(n+2)n!}{2^n(n+1)(n+1)!} \\ &= \frac{2(n+2)}{(n+1)^2} \\ &= \frac{2n+4}{n^2+2n+1} \end{aligned}$$

so, using L'Hopital's rule,

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{2n+4}{n^2+2n+1} \\ &= \lim_{n\to\infty} \frac{2}{2n+2} = \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0 \end{aligned}$$

so the sequence converges by the Ratio Test.

7. (15 points)

Find the limit of this sequence.

$$\lim_{n\to\infty} \frac{2n^2+1}{\sqrt{3n^4+1}}$$

We will divide the numerator and denominator each by n^2. This means dividing by n^4 inside the square root sign. Thus we get

$$\begin{aligned} \lim_{n\to\infty} \frac{2n^2+1}{\sqrt{3n^4+1}} &= \lim_{n\to\infty} \frac{2+1/n^2}{\sqrt{3+1/n^4}} \\ &= \frac{\lim_{n\to\infty}(2+1/n^2)}{\lim_{n\to\infty}\sqrt{(3+1/n^4)}} \\ &= \frac{2}{\sqrt{3}} \end{aligned}$$