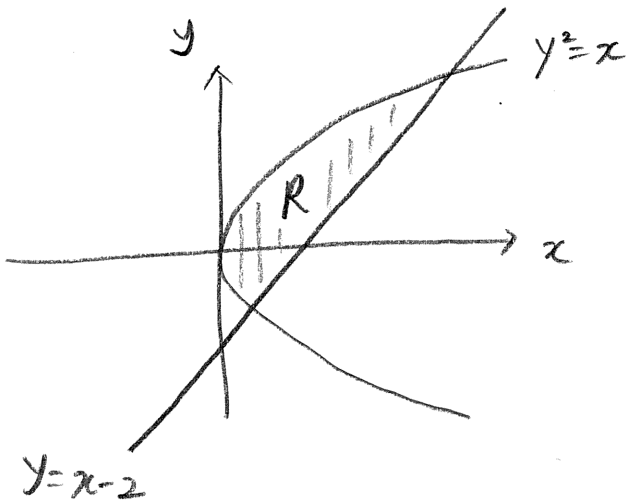
1. (10 points) Find the area in the region between the curves $y^2 = x$ and $y = x - 2$.



① Find intersecting pts : $y^2 = y + 2$

$\Leftrightarrow y^2 - y - 2 = 0$

$\Leftrightarrow (y-2)(y+1) = 0$

$\Leftrightarrow y = -1, 2$

② Area of $R = \int_{-1}^{2} (y+2) - y^2 \, dy$

$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= 8 - \frac{1}{2} - 3$

$= 4.5$

2. (15 points) Find the average value of $f(x) = \sec^4 x \tan^3 x$ on the interval $[0, \pi/4]$.

$$\text{Average} = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \sec^4 x \tan^3 x \, dx$$

$$\int_0^{\frac{\pi}{4}} \sec^4 x \tan^3 x \, dx = \int_0^{\frac{\pi}{4}} \sec^2 x \,(1+\tan^2 x)\tan^3 x \, dx$$

$$\left(\text{Let } \begin{cases} u = \tan x \\ du = \sec^2 x \, dx \end{cases} . \quad \begin{cases} x = 0 & u = \tan 0 = 0 \\ x = \frac{\pi}{4} & u = \tan\frac{\pi}{4} = 1 \end{cases} \right)$$

$$= \int_0^1 (1+u^2)\,u^3 \, du$$

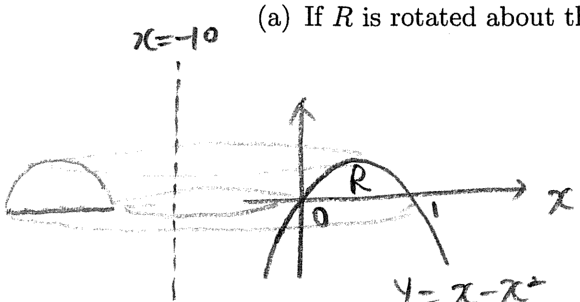
$$= \int_0^1 u^3 + u^5 \, du$$

$$= \left[\frac{u^4}{4} + \frac{u^6}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}.$$

So, the average $= \frac{4}{\pi} \cdot \frac{5}{12} = \frac{5}{3\pi}$

3. (15 points) All of the following problems concern the solid of revolution generated by rotating about a given axis the region R, which lies between the x-axis and the curve $y = x - x^2$. You may use either the method of disks/washers or the method of cylindrical shells, but you **must clearly indicate which method** you are using in each problem.

(a) If R is rotated about the line $x = -10$, compute the volume of the resulting solid.



shell method:

$$\text{Volume} = \int_0^1 2\pi\,(x-(-10))\,(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (x+10)\,(x-x^2)\,dx$$

$$= 2\pi \int_0^1 x^2 - x^3 + 10x - 10x^2\,dx$$

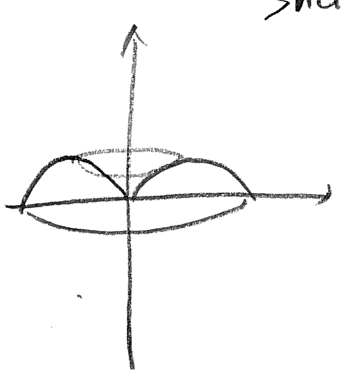
$$= 2\pi \int_0^1 -x^3 - 9x^2 + 10x\,dx$$

$$= 2\pi \left[-\frac{x^4}{4} - 3x^3 + 5x^2\right]_0^1$$

$$= 2\pi\left(-\frac{1}{4} - 3 + 5\right) = \frac{7}{2}\pi$$

(b) If R is rotated about the y-axis, set up but do not evaluate an integral for computing the volume of the resulting solid.

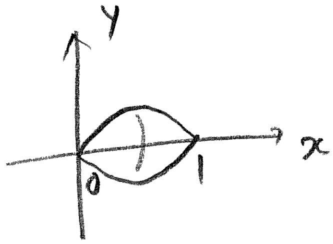
shell method:



$$\int_0^1 2\pi\, x\,(x-x^2)\,dx$$

4. (15 points) All of the following problems concern the solid of revolution generated by rotating about a given axis the region R, which lies between the x-axis and the curve $y = x - x^2$. You may use either the method of disks/washers or the method of cylindrical shells, but you **must clearly indicate which method** you are using in each problem.

(a) If R is rotated about the x-axis, compute the volume of the resulting solid.



disk method:

$$\text{Volume} = \int_0^1 \pi(x-x^2)^2\,dx$$

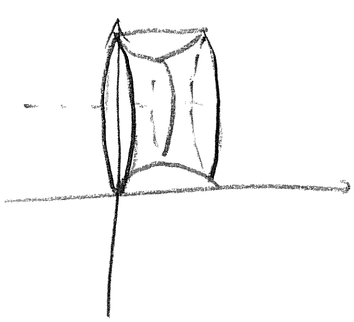
$$= \int_0^1 \pi\,(x^2-2x^3+x^4)\,dx$$

$$= \pi\left[\frac{x^3}{3}-\frac{1}{2}x^4+\frac{x^5}{5}\right]_0^1$$

$$= \pi\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)$$

$$= \pi\left(\frac{1}{30}\right) = \frac{\pi}{30}$$

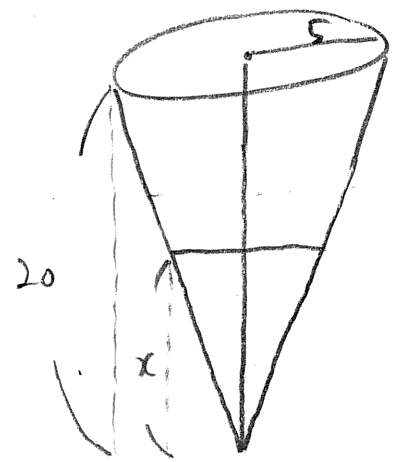
(b) If R is rotated about the line $y = 3$, set up but do not evaluate an integral for computing the volume of the resulting solid.



disk method:

$$\text{Volume} = \int_0^1 \pi\left(3^2-(3-(x-x^2))^2\right)dx$$

5. (15 points) Assume that the density of a certain liquid is $D\ kg/m^3$ and that the acceleration due to gravity on Planet X is $G\ m/s^2$. A tank in the shape of an inverted cone has a height of 20 meters and a base radius of 5 meters and is filled with that liquid to a depth of 15 meters. Set up but **do not evaluate** an integral to determine the amount of work needed to pump all of the water to the top of the tank on Planet X.



$$\text{Work} = GD\int_0^{15} \underbrace{(20-x)}_{\text{height}}\ \underbrace{\pi r^2\,dx}_{\text{volume}}$$

$$\left(\frac{r}{x} = \frac{5}{20} \iff r = \frac{1}{4}x \right)$$

$$= GD\int_0^{15} (20-x)\,\pi\left(\frac{x}{4}\right)^2 dx$$

6. (15 points) Find the following indefinite integrals.

(a)

$$\int x \sin(x) \cos(x)\, dx$$

$$= \frac{1}{2}\int x \sin(2x)\, dx$$

$$= \frac{1}{2}\left(x\left(-\frac{\cos(2x)}{2}\right) + \int \frac{\cos 2x}{2}\, dx \right)$$

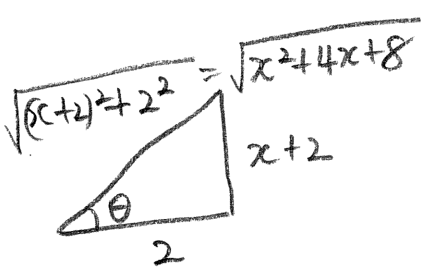
$$= \frac{1}{2}\left(-\frac{x\cos(2x)}{2} + \frac{1}{4}\sin(2x) + C_1 \right)$$

$$= -\frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x) + C$$

(b)

$$\int \frac{1}{\sqrt{x^2+4x+8}}\, dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2+4}}\, dx \qquad \left(\begin{array}{l} x+2 = 2\tan\theta \\ dx = 2\sec^2\theta\, d\theta \end{array} \right.$$



$$= \int \frac{1}{\sqrt{4(\tan^2\theta+1)}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{4\sec^2\theta}}\, 2\sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left| \frac{\sqrt{x^2+4x+8}}{2} + \frac{x+2}{2} \right| + c$$

7. **(15 points)**

(a) Find the following indefinite integral.

$$\int \frac{e^x}{e^{2x}+e^x-2}\,dx$$

Let $\begin{cases} u = e^x \\ du = e^x dx. \end{cases}$ Then

$$= \int \frac{du}{u^2+u-2} = \int \frac{du}{(u+2)(u-1)}.$$

$$\frac{1}{(u+2)(u-1)} = \frac{A}{u+2} + \frac{B}{u-1}$$

$$= \frac{A(u-1)+B(u+2)}{(u+2)(u-1)}.$$

So, $1 = A(u-1)+B(u+2)$.

Plugging $\begin{cases} u=1 \text{ in}, & B = \frac{1}{3} \\ u=-2 \text{ in}, & A = -\frac{1}{3}. \end{cases}$

$$= -\frac{1}{3}\int \frac{du}{u+2} + \frac{1}{3}\frac{du}{u-1}$$

$$= -\frac{1}{3}\ln|u+2| + \frac{1}{3}\ln|u-1| + c$$

$$= -\frac{1}{3}\ln|e^x+2| + \frac{1}{3}\ln|e^x-1| + C.$$

(b) Set up the partial fraction decomposition for the following integral in terms of variables, but do not solve for those variables.

$$\int \frac{1}{(x^3+x^2+x)(x^3-x)(x^2+x)(x^2+x+1)}\,dx$$

$$\begin{cases} x^3+x^2+x = x(x^2+x+1) \\ x^3-x = x(x^2-1) = x(x-1)(x+1) \\ x^2+x = x(x+1) \end{cases}$$

for x^2+x+1, $b^2-4ac = 1-4 = -3 < 0$

$\Rightarrow$ no more decomposition possible

$$= \int \frac{1}{x^3(x-1)(x+1)^2(x^2+x+1)^2}\,dx$$

$$= \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{(x^2+x+1)^2}\,dx$$