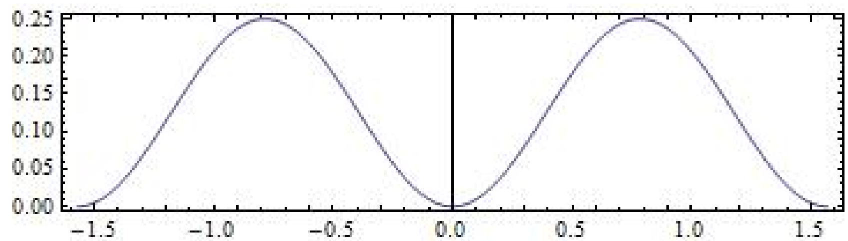
c

Math 162: Calculus IIA

First Midterm Exam ANSWERS

February 27, 2019

1. (20 points) Find the average value of the function $f(x) = \sin^2(x)\cos^2(x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Answer:

$$\begin{aligned} f_{ave} &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos^2(x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)\,dx \\ &= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} (1-\cos^2(2x))\,dx \\ &= \frac{1}{8\pi}\int_{-\pi/2}^{\pi/2} (1-\cos(4x))\,dx \\ &= \frac{1}{8\pi}\left[x - \frac{1}{4}\sin(4x)\right]_{-\pi/2}^{\pi/2} = \frac{1}{8} \end{aligned}$$

2. (20 points) If $a \neq 0$, evaluate

$$\int \cos^3(ax+b)\,dx$$

in terms of a and b.

Answer:

Let $u = \sin(ax+b)$. Then $du = a\cos(ax+b)dx$, so

$$\begin{aligned}\int \cos^3(ax+b)dx &= \int (1-\sin^2(ax+b))\cos(ax+b)dx \\ &= (1/a)\int(1-u^2)du \\ &= (1/a)(u-u^3/3) \\ &= (1/a)(\sin(ax+b)-(1/3)\sin^3(ax+b)).\end{aligned}$$

3. (20 points) A heavy rope, 20 m long, weighs 2 kg/m and hangs over the edge of a building 100 m high. Consider that one ties a heavy ball at the end of this rope with weight 20 kg. How much work is done in pulling half the rope to the top of the building?

Answer:

The rope requires work

$$\int_0^{10}(20-x)\times 2g dx = 40xg - x^2 g\Big]_0^{10} = 300gJ.$$

The ball requires work $20 \times 10g = 200g$ J. Thus, the total work is $500gJ$.

You can specialize g by $9.8m/s^2$ or just leave it.

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int x^n e^x\,dx \qquad \text{in terms of} \qquad \int x^{n-1}e^x\,dx$$

Answer:

Let $u = x^n$ and $dv = e^x dx$, so $du = nx^{n-1}$ and $v = e^x$. Then we have

$$\int x^n e^x\,dx \;=\; \int u\,dv \;=\; uv - \int v\,du$$

$$= \quad x^n e^x - n \int x^{n-1} e^x \, dx.$$

(b) (10 points) Use this formula to find

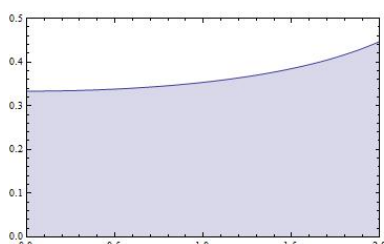
$$\int x^3 e^x \, dx.$$

Answer:

$$\begin{aligned}
\int x^3 e^x \, dx &= x^3 e^x - 3 \int x^2 e^x \, dx \\
&= x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x \, dx. \right) \\
&= (x^3 - 3x^2) e^x + 6 \int x e^x \, dx \\
&= (x^3 - 3x^2) e^x + 6 \left(x e^x - \int e^x \, dx \right) \\
&= (x^3 - 3x^2 + 6x) e^x - 6 \int e^x \, dx \\
&= (x^3 - 3x^2 + 6x - 6) e^x + C
\end{aligned}$$

5. (20 points)

(a) (10 points) Find the volume of the solid obtained by rotating the region bounded by the x-axis and the curve $y = 1/\sqrt{9 - x^2}$ for $0 \leq x \leq 2$



about the y-axis.

Answer:

Using the cylindrical shell method, we will integrate with respect to x, and have a radius of

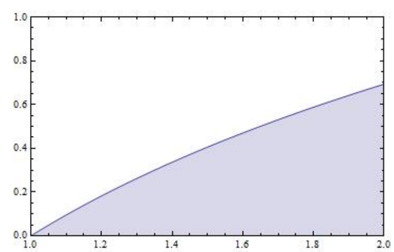
$r = x$, height of $h = \frac{1}{\sqrt{9-x^2}}$, and bounds of $x = 0$ to $x = 2$. Thus

$$V = 2\pi \int_0^2 \frac{x}{\sqrt{9-x^2}} dx$$

Using a u-substitution of $u = 9 - x^2$, $du = -2xdx$ gives

$$\begin{aligned} V &= \int_0^2 \frac{2\pi x}{\sqrt{9-x^2}} dx = -\int_9^5 \frac{\pi}{\sqrt{u}} du \\ &= -2\pi\sqrt{u}\Big|_9^5 \\ &= 2\pi\sqrt{9} - 2\pi\sqrt{5} = 6\pi - 2\pi\sqrt{5} \end{aligned}$$

(b) (10 points) Find the volume of the solid obtained by rotating the region bounded by the x-axis and the curve $y = ln(x)$ for $1 \le x \le 2$ about the line $x = -1$.



Answer:

Using the washer method, we will integrate with respect to y, and have an inner radius of $1 + e^y$, an outer radius of 3, and bounds of $y = 0$ to $y = ln(2)$. Thus

$$\begin{aligned} V &= \pi \int_0^{ln(2)} [(3)^2 - (1 + e^y)^2] dy \\ &= \pi \int_0^{ln(2)} [9 - (1 + 2e^y + e^{2y})] dy \\ &= \pi \int_0^{ln(2)} (8 - 2e^y - e^{2y}) dy \\ &= \pi(8y - 2e^y - \frac{1}{2}e^{2y})\Big|_0^{ln(2)} \\ &= \pi[(8ln(2) - 2e^{ln(2)} - \frac{1}{2}e^{2ln(2)}) - (0 - 2e^0 - \frac{1}{2}e^0)] \\ &= \pi(8ln(2) - 4 - 2 + 2 + \frac{1}{2}) = \pi(8ln(2) - \frac{7}{2}) \end{aligned}$$