Math 162: Calculus IIA

Midterm I

February 23rd, 2016

Please circle your section:

Gage MW 2pm **Harper TR 9:40am** **Lubkin MWF 9am**

Lungstrum MW 3:25pm **Neuman TR 4:50pm** **Tucker MW 10:25am**

NAME (please print legibly): SOLUTIONS

Your University ID Number: __________

Your University email __________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: __________

QUESTION	VALUE	SCORE
1	12	
2	12	
3	10	
4	13	
5	13	
6	13	
7	15	
8	12	
TOTAL	100	

Instructions:

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You must be physically separated from your cell phone.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 10 pages.**

Formulas:

- $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\int \tan(x)dx = \ln|\sec(x)| + C$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

1. (12 points) Find the area contained between the curves $y = x^3$ and $y = 4x$.

$$x^3 = 4x \Rightarrow x(x^2-4) = 0 \Rightarrow x(x+2)(x-2) = 0$$

$$\Rightarrow x = -2, 0, 2$$

$$\int_{-2}^{2} |x^3 - 4x| \, dx = \int_{-2}^{0} x^3 - 4x \, dx + \int_{0}^{2} 4x - x^3 \, dx$$

$$= \frac{x^4}{4} - 2x^2 \Big]_{x=-2}^{x=0} + 2x^2 - \frac{x^4}{4} \Big]_{x=0}^{x=2}$$

$$= \left[(0-0) - \left(\frac{16}{4} - 8\right)\right] + \left[\left(8 - \frac{16}{4}\right) - (0-0)\right]$$

$$= 8$$

2. (12 points) A spring has a natural length of 2m. A force of 30N is needed to stretch the spring to a length of 3m.

(a) Find the work, in Joules, required to stretch the spring from 3m to 4m.

$F(x) = kx$

$30N = k\,(1m)$

$\Rightarrow 30 = k$

$$\int_3^4 30x\,dx$$

$$= 15x^2 \Big]_{x=3}^{x=4}$$

$$= 15\,(16-9)$$

$$= 105$$

(b) Now find the work, in Joules, required to stretch the spring from 3m to (3+c)m, where c is a positive constant.

$$\int_3^{(3+c)} 30x\,dx = 15x^2 \Big]_{x=3}^{x=3+c} = 15\left((3+c)^2 - 3^2\right)$$

3. **(10 points)** Find the area underneath the curve $y = e^{\sqrt{x}}$ from $x = 0$ to $x = 1$.

$$\int_0^1 e^{\sqrt{x}}\, dx \overset{\text{substitution}}{=} \int_0^1 2u e^u\, du$$

$u^2 = x$

$2u\, du = dx$

$w = 2u$ $\qquad dv = e^u\, du$

$dw = 2\, du$ $\qquad v = e^u$

$$= 2ue^u \Big]_{u=0}^{u=1} - \int_0^1 2e^u\, du$$

$$= 2e - \left(2e^u \Big]_{u=0}^{u=1} \right)$$

$$= 2e - (2e - 2)$$

$$= 2$$

4. **(13 points)** The region between the x-axis and the curve $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the x-axis. Compute the volume.

$$\int_0^\pi \pi \left(\sin(x)\right)^2 dx = \pi \int_0^\pi \frac{1 - \cos(2\theta)}{2} \, d\theta$$

$$= \frac{\pi}{2} \int_0^\pi 1 - \cos(2\theta) \, d\theta$$

$$= \frac{\pi}{2} \left(\theta - \frac{\sin(2\theta)}{2} \Big]_{\theta=0}^{\theta=\pi} \right)$$

$$= \frac{\pi}{2} \left((\pi - 0) - (0 - 0) \right)$$

$$= \frac{\pi^2}{2}$$

5. (13 points) The region between the x-axis and the curve $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the line $x = -1$. Compute the volume.

$$\int_0^{\pi} 2\pi(1+x)\sin(x)\,dx = 2\pi\left(\int_0^{\pi} \sin(x)\,dx + \int_0^{\pi} x\sin(x)\,dx\right)$$

$$= 2\pi\left(-\cos(x)\Big]_{x=0}^{x=\pi} + \left(-x\cos(x)\right)\Big]_{x=0}^{x=\pi} + \int_0^{\pi} \cos(x)\,dx\right)$$

$$= 2\pi\left(2 + \pi + 0\right)$$

$$= 4\pi + 2\pi^2$$

6. (13 points) Find the following definite integrals.

(a)

$$\int_0^{\pi/4} \sec^4(\theta)\tan^4(\theta)\,d\theta$$

$$\int_0^{\frac{\pi}{4}} (1+\tan^2\theta)\tan^4\theta \quad \sec^2\theta\, d\theta$$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$$= \int_0^1 (1+u^2)u^4\, du = \int_0^1 u^4 + u^6\, du$$

$$= \frac{u^5}{5} + \frac{u^7}{7}\Big]_{u=0}^{u=1} = \frac{1}{5} + \frac{1}{7}$$

(b)

$$\int_0^{\pi} x^2\cos(x)dx$$

$u_1 = x^2 \qquad dv_1 = \cos(x)\,dx$

$du_1 = 2x\,dx \qquad v_1 = \sin(x)$

$$x^2\sin(x)\Big]_{x=0}^{x=\pi} - \int_0^{\pi} 2x\sin(x)\,dx =$$

$$= 0 - \int_0^{\pi} 2x\sin(x)\,dx \quad = -\left(-2x\cos(x)\Big]_{x=0}^{x=\pi} + \int_0^{\pi} 2\cos(x)\,dx\right)$$

$u_2 = 2x \qquad dv_2 = \sin(x)\,dx$

$du_2 = 2\,dx \qquad v_2 = -\cos(x)$

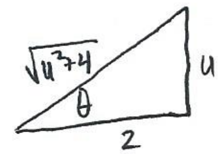
$$= -(2\pi + 0)$$

$$= -2\pi$$

7. (15 points) Find the following indefinite integrals.

(a)

$$\int \frac{dx}{\sqrt{x^2 - 6x + 13}}$$



$$\int \frac{dx}{\sqrt{(x-3)^2+4}} \xrightarrow[1]{\substack{u = x-3 \\ du = dx}} \int \frac{du}{\sqrt{u^2+4}}$$

$u = 2\tan\theta \implies \frac{u}{2} = \tan\theta$

$du = 2\sec^2\theta \, d\theta$

$$\xrightarrow{2} \int \frac{1}{2\sec\theta} \cdot 2\sec^2\theta \, d\theta = \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C = \ln\left|\frac{\sqrt{(x-3)^2+4} + (x-3)}{2}\right| + C$$

(b)

$$\int \frac{dx}{\sqrt{-x^2 + 6x - 5}}$$

$$= \int \frac{dx}{\sqrt{4-(x-3)^2}} \xrightarrow{\substack{x-3 = 2\sin\theta \\ dx = 2\cos\theta \, d\theta}} \int \frac{1}{\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta} \, d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x-3}{2}\right) + C$$

8. (12 points)

(a) Find the following indefinite integral.

$$\int \frac{x^2 - 2x - 1}{(x-1)(x^2+1)}\, dx$$

Partial Fraction:

$$\frac{x^2-2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2-2x-1 = A(x^2+1) + (Bx+C)(x-1)$$

$\underline{x=1}$

$-2 = 2A$

$-1 = A$

$\underline{x=0}$

$-1 = -1 + (-C)$

$C = 0$

$\underline{x=-1}$

$2 = -2 + (-2)B$

$-2 = B$

$$= \int \frac{-1}{x-1} + \frac{-2x}{x^2+1}\, dx$$

$$= \ln\left|\frac{1}{x-1}\right| + \ln\left(\frac{1}{x^2+1}\right) + C$$

(b) Set up the partial fraction decomposition for the following integral in terms of variables, but do not solve for those variables.

$$\int \frac{x^2 - 5x + 16}{(x^2+1)^2(x^2-1)^2(x-1)^2}\, dx$$

$$\frac{x^2-5x+16}{(x^2+1)^2(x+1)^2\underbrace{(x-1)^2(x-1)^2}_{(x-1)^4}} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{I}{(x-1)^3} + \frac{J}{(x-1)^4}$$