MIDTERM 1 REVIEW SHEET

1. THE SUBSTITUTION RULE

◇ If $u = g(x)$ is a differentiable function and f is continuous on the range of g, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

◇ In the case of definite integrals, if $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

◇ Suppose f is continuous on $[-a, a]$.

▷ If f is even, i.e., $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

▷ If f is odd, i.e., $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.

2. AREAS BETWEEN CURVES

◇ The area of the region bounded by $f(x)$, $g(x)$, $x = a$, and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ on the interval $[a, b]$, is given by

$$A = \int_a^b [f(x) - g(x)]\,dx$$

▷ You can remember this formula as "top curve minus bottom curve". Keep in mind that the top curve and the bottom curve may change over the interval. For example, this happens with the region bounded by $y = \sin x$ and $y = \cos x$ over the interval $[0, \frac{\pi}{2}]$.

◇ The area of the region bounded by $f(y)$, $g(y)$, $y = c$, and $y = d$, where f and g are continuous functions and $f(y) \geq g(y)$ on the interval $[c, d]$, is given by

$$A = \int_c^d [f(y) - g(y)]\,dy$$

▷ You can remember this formula as "right curve minus left curve". Again, keep in mind that the right curve and the left curve may change over the interval. This is why drawing a picture of the region is always a good idea.

3. VOLUMES

◇ Suppose you have a solid S lying between $x = a$ and $x = b$ and you want to compute its volume. Let's say that the area of a cross-section at x taken perpendicular to the x-axis is given by a continuous function $A(x)$. Then the volume of S is given by

$$V = \int_a^b A(x)\,dx$$

◇ Suppose you have a solid S lying between $y = c$ and $y = d$ and you want to compute its volume. Let's say that the area of a cross-section at y taken perpendicular to the y-axis is given by a continuous function $A(y)$. Then the volume of S is given by

$$V = \int_c^d A(y)\,dy$$

◇ In the case of solids of revolution, the solid is obtained by rotating a region around some line. When cross-sections are taken *perpendicular* to this line, they are disks or washers. In the first case, the area is given by πR^2 where R is the radius of a typical disk. In the second case, the area is $\pi(R^2 - r^2)$ where R and r are the outer and inner radii of the washer, respectively.

4. VOLUMES BY CYLINDRICAL SHELLS

◇ In the case of solids of revolution, the solid is obtained by rotating a region around some line. When cross-sections are taken *parallel* to this line, they are cylindrical shells.

▷ If the region is obtained from rotating about a vertical line, the shells are upright. Here, you look at a typical shell at some point x and find expressions for it's radius and height, say r_x and h_x. The volume is then found by computing

$$V = \int_a^b 2\pi r_x h_x\,dx$$

▷ If the region is obtained from rotating about a horizontal line, the shells are on their side. Here, you look at a typical shell at some point y and find expressions for it's radius and height, say r_y and h_y. The volume is then found by computing

$$V = \int_c^d 2\pi r_y h_y\,dy$$

5. WORK

◇ Remember that Force = mass $*$ acceleration ($F = ma$). If the acceleration is constant, the Force is constant, and in this case Work = Force $*$ distance ($W = Fd$).

◇ If an object moves along the x-axis from $x = a$ to $x = b$, and at each point a Force of $f(x)$ acts on the object, the total Work done in moving from $x = a$ to $x = b$ is

$$W = \int_a^b f(x)\,dx$$

◇ *Hooke's Law*: The force required to hold a spring at x meters (or feet) past its natural length is proportional to x, i.e., $f(x) = kx$ where $k > 0$ is the spring constant. Here, the Work done in stretching the spring from a meters (or feet) past its natural length to b meters (or feet) past its natural length is given by

$$W = \int_a^b kx\,dx$$

- ▷ NOTE: Remember that in these types of questions, the units should be in meters or feet (not centimeters or inches). Also, all distances (e.g., the x, a, and b above) are distances *past the natural length*.

◇ A popular type of question involves emptying liquids from containers of various shapes. There are two cases, depending on whether lengths are measured in meters or feet. In both cases, you should start by finding the volume of a typical "slice".

- ▷ If lengths are in meters, follow these steps, where each arrow signifies multiplication by the thing above it (except the last one which means integrate W_i):

$$V_i \xrightarrow{\text{density}} m_i \xrightarrow{\text{gravity}} F_i \xrightarrow{\text{distance}} W_i \xrightarrow{\text{Integrate}} W$$

- ▷ If lengths are in feet, a similar method is used:

$$V_i \xrightarrow{\text{density}} F_i \xrightarrow{\text{distance}} W_i \xrightarrow{\text{Integrate}} W$$

6. INTEGRATION BY PARTS

◇ The formula is given by $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$, or letting $u = f(x)$ and $v = g(x)$, we obtain the easy to remember formula $\int u\,dv = uv - \int v\,du$.

◇ Using the formula requires choosing something to be u and something to be dv. Here are some pointers:

- ▷ Powers of x are good to choose for u (e.g., $\int x^2 \sin x\,dx$ choose $u = x^2$).
- ▷ If there is a term you don't know how to easily integrate, you can't choose this to be dv, so it has to be chosen for u (e.g., $\int \ln x\,dx$ choose $u = \ln x$).
- ▷ You may have to use integration by parts more than once to get the answer (e.g., $\int x^2 e^x\,dx$).
- ▷ Remember the "trick" involved in integrating things like $\int e^x \sin x\,dx$ or $\int e^x \cos x\,dx$.

◇ When evaluating definite integrals, the formula becomes

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx$$

7. TRIGONOMETRIC INTEGRALS

◇ Integrals of the form $\int \sin^m x \cos^n x\,dx$

- ▷ Case 1: m odd. Split off one factor of $\sin x$, use the identity $\sin^2 x = 1 - \cos^2 x$ to convert the remaining powers of sine to cosine, and make the substitution $u = \cos x$.
- ▷ Case 2: n odd. Split off one factor of $\cos x$, use the identity $\cos^2 x = 1 - \sin^2 x$ to convert the remaining powers of cosine to sine, and make the substitution $u = \sin x$.
- ▷ Case 3: m and n are both odd. Use the method of either Case 1 or Case 2 (Case 2 is easier).
- ▷ Case 4: m and n are both even. Here, you just use the following identities to reduce the powers until you get something that you know how to integrate:

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = \frac{1}{2}[1 - \cos 2x]$$

$$\cos^2 x = \frac{1}{2}[1 + \cos 2x] \qquad \sin x \cos x = \frac{1}{2}\sin 2x$$

◇ Integrals of the form $\int \tan^m x \sec^n x\,dx$

- ▷ Case 1: m odd and $n \geq 1$. Split off one factor of $\sec x \tan x$, use the identity $\tan^2 x = \sec^2 x - 1$ to convert the remaining powers of tangent to secant, and make the substitution $u = \sec x$.
- ▷ Case 2: n even. Split off one factor of $\sec^2 x$, use the identity $\sec^2 x = 1 + \tan^2 x$ to convert the remaining powers of secant to tangent, and make the substitution $u = \tan x$.
- ▷ Other cases don't really have a clear-cut method. Using trigonometric identities and the following results will usually help:

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \text{or} \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

◇ For integrals of the form $\int \sin mx \cos nx\,dx$, use the following identities:

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

8. TRIGONOMETRIC SUBSTITUTION

- ◇ In these types of problems, you will see a term involving $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$, where $a > 0$ is some number. Typically, you will see the square root of one of these terms, but in some problems, you may the term raised to a different power (other than $\frac{1}{2}$).

- ◇ In any case, the table below tells you how to handle these types of integrals.

EXPRESSION	SUBSTITUTION	IDENTITY	RESULT
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$	$a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$a\tan\theta$

- ◇ Always remember to substitute for each term in the integral, including the dx!