Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 3, 2024

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

1. (20 points) Let $c > 0$ be a fixed number. Find the area between the two curves $x = cy$ and $x = cy^2$. Note that your answer will depend on c.

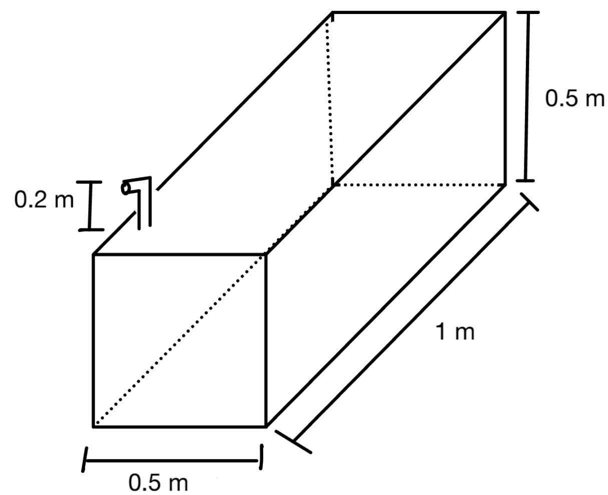
Answer:

Our curves are of the form $x = f(y)$ and $x = g(y)$; so it makes sense to integrate with respect to y. To find the bounds, $cy = x = cy^2$ implies $y = 1$ or 0. Sketching the region in question shows that $x = cy$ will be the right-most curve. [Alternatively: For $0 \leq y \leq 1$, $y^2 \leq y$ so that, as $c > 0$, we have $cy^2 \leq cy$.] So

$$\begin{aligned}
\text{Area} &= \int_0^1 (\text{right} - \text{left})\ dy \\
&= \int_{-1}^1 (cy - cy^2)\ dy \\
&= c\int_0^1 (y - y^2)\ dy \\
&= c\left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Bigg]_{y=0}^{1} \\
&= c\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right) \\
&= \frac{c}{6}
\end{aligned}$$

2. (20 points) This is a work problem in metric units. A fish tank in the shape of a rectangular prism has dimensions 0.5 m $\times$ 0.5 m $\times$ 1 m, as pictured below.

There is a spout on top of the fish tank whose outlet is 0.2 meters above the top of the tank. Suppose the tank is filled with water to a depth of 0.25 meters. How much work does it take to pump all of the water out through the spout? Use $\rho = 1,000$ kilograms per meter cubed for the density of water, and use $g = 10$ meters per second squared for the acceleration due to gravity.



Answer:

Place an x-axis along the vertical dimension of the tank. Each cross section of the tank will be a rectangle of dimensions 0.5 meters by 1 meter. Thus the volume of a thin slice of the tank is $0.5\ dx$ m^3. The weight of any thin layer of water is then

$$\text{weight} = (\text{mass})g = (\text{density})(\text{volume})g = \rho g 0.5\ dx.$$

Each layer of water must travel from x meters above the base of the tank to $0.5 + 0.2$ meters above the base of the tank. Thus the distance each layer tavels is $0.7 - x$. So the work done to move a thin layer of water is

$$\text{work to move thin layer of water} = (\text{distance})(\text{weight}) = (0.7 - x)(0.5\rho g)\ dx$$

Since the tank is filled up to a depth of 0.25 meters, then the total work done in pumping all the water out of the tank is

$$\begin{aligned} W &= \int_0^{0.25} 0.5\rho g(0.7 - x)dx \\ &= \frac{\rho g}{2}(0.7x - 0.5x^2)\Big]_{x=0}^{0.25} \\ &= \frac{\rho g}{2}((0.7)(0.25) - 0.5(0.25)^2) \\ &= \frac{10,000}{2}0.25(0.7 - 0.125) \\ &= \frac{10,000}{8}(0.575) \\ &= \frac{5750}{8} \quad \text{Joules.} \end{aligned}$$

3. (20 points) (a) (15 points) Let $n \geq 2$ be an integer. Find numbers A_1, A_2, and A_3 so that

$$\int x^n \cos(x)\, dx = A_1 x^n \sin(x) + A_2 x^{n-1} \cos(x) + A_3 \int x^{n-2} \cos(x)\, dx.$$

[Hint: Integrate $x^n \cos(x)$ by parts twice. Note that some of the numbers A_1, A_2, and A_3 could depend on n.]

Answer:

Let $u = x^n$ and $dv = \cos(x)dx$ so that $du = nx^{n-1}dx$ and $v = \sin(x)$. Integrating by parts once gives

$$\int x^n \cos(x)\, dx = uv - \int v\, du = x^n \sin(x) - \int nx^{n-1} \sin(x)\, dx.$$

To integrate $\int x^{n-1} \sin(x)\, dx$, take $w = x^{n-1}$, $dz = \sin(x)dx$ so that $dw = (n-1)x^{n-2}$ and $z = -\cos(x)$. Then

$$\begin{aligned} \int x^{n-1} \sin(x)\, dx &= -x^{n-1} \cos(x) - \int (-\cos(x))(n-1)x^{n-2}\, dx \\ &= -x^{n-1} \cos(x) + (n-1) \int x^{n-2} \cos(x)\, dx. \end{aligned}$$

Therefore

$$\begin{aligned} \int x^n \cos(x)\, dx &= x^n \sin(x) - n \int x^{n-1} \sin(x)\, dx \\ &= x^n \sin(x) - n\Big(- x^{n-1} \cos(x) + (n-1) \int x^{n-2} \cos(x)\, dx \Big) \\ &= x^n \sin(x) + nx^{n-1} \cos(x) - n(n-1) \int x^{n-2} \cos(x)\, dx. \end{aligned}$$

So $A_1 = 1$, $A_2 = n$ and $A_3 = -n(n-1)$.

(b) (5 points) Evaluate

$$\int_0^\pi x^2 \cos(x)\ dx.$$

Note: If you apply the formula from part (a) and your values for A_1, A_2, and A_3 are incorrect, you may not receive credit for this problem.

Answer:

By the formula in part (a),

$$\int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2(2-1) \int \cos(x)\ dx$$

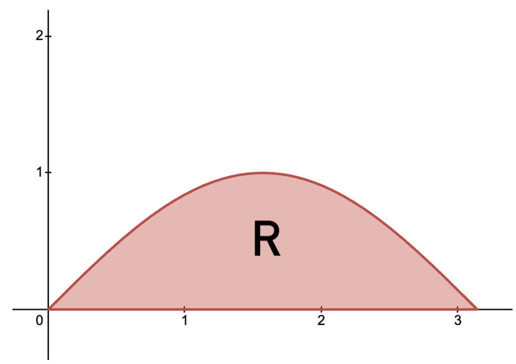
so that

$$\int x^2 \cos(x)\ dx = x^2 \sin(x) + 2x \cos(x) - 2\sin(x) + C.$$

Therefore, using the fact that $\sin(0) = \sin(\pi) = 0$, we get

$$\begin{aligned}
\int_0^\pi x^2 \cos(x)\ dx &= x^2 \sin(x) + 2x \cos(x) - 2\sin(x) \Big]_{x=0}^{\pi} \\
&= 2x \cos(x) \Big]_{x=0}^{\pi} \\
&= 2\pi \cos(\pi) - 2(0) \cos(0) \\
&= 2\pi(-1) \\
&= -2\pi.
\end{aligned}$$

4. (20 points) Let R be the space between the graph of $f(x) = \sin(x)$ and the x-axis, which lies in between the vertical lines $x = 0$ and $x = \pi$. Find the volume of the solid generated when rotating the region R about the horizontal line $y = 2$.



Answer:

The method of washers will be applied. The inner radius, $r(x)$ will be given by the distance between the top of the graph and the line $y = 2$. This is equal to

$$r(x) = 2 - \sin(x)$$

The outer radius, $R(x)$, is constant, given by the distance between the x axis and the line $y = 2$. This is equal to

$$R(x) = 2$$

The infinitesimal cross-sectional area will be given by

$$dA = \pi(R(x)^2 - r(x)^2)dx$$

Given that $0 \leq x \leq \pi$, the volume is then given by

$$V = \int_0^\pi \pi(R(x)^2 - r(x)^2)dx = \pi \int_0^\pi 4 - (2 - \sin(x))^2 dx$$
$$= \pi \int_0^\pi 4 - (4 - 4\sin(x) + \sin^2(x))dx = \pi \int_0^\pi 4\sin(x) - \sin^2(x)dx$$

Applying the trig identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, and using the fact that $\int_0^\pi \sin(x)dx = 2$, we get that

$$V = 8\pi - \frac{\pi}{2} \int_0^\pi 1 - \cos(2x)dx$$

The antiderivative of $\cos(2x)$ is $\frac{1}{2}\sin(2x)$, so

$$V = 8\pi - \frac{\pi}{2}\left(x - \frac{\sin(2x)}{2}\right)\Big|_0^{\pi} = 4\pi - \frac{\pi}{2}(\pi - 0) + \frac{\pi}{2}(0 - 0) = 8\pi - \frac{\pi^2}{2}$$

5. (20 points) Evaluate

$$\int \cos^5(2x+1)\,dx.$$

Answer:

Let $u = \sin(2x+1)$.

Then $du = 2\cos(2x+1)dx$, so

$$\begin{aligned}
\int \cos^5(2x+1)dx &= \int (1-\sin^2(2x+1))^2\cos(2x+1)dx = \frac{1}{2}\int (1-u^2)^2 du \\
&= \frac{1}{2}\int (1-2u^2+u^4)du \\
&= \frac{1}{2}\left(u - \frac{2u^3}{3} + \frac{u^5}{u}\right) + C \\
&= \frac{u}{2} - \frac{u^3}{3} + \frac{u^5}{10} + C \\
&= \frac{\sin(2x+1)}{2} - \frac{\sin^3(2x+1)}{3} + \frac{\sin^5(2x+1)}{10} + C.
\end{aligned}$$

Scratch paper

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