Math 162: Calculus IIA

First Midterm Exam ANSWERS

September 24, 2023

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution (known in Doug's section as *the rabbit trick.*)) for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

1. (20 points)

(a) (10 points) Find the integral

$$\int \frac{x}{\sqrt{x^2-6x+13}}\,dx$$

Answer:

First, complete the square:

$$\begin{aligned} x^2-6x+13 &= x^2-6x+9+4 \\ &= (x-3)^2+4 \end{aligned}$$

We then want the substitution $x-3=2\tan(\theta)$, so $dx = 2\sec^2(\theta)d\theta$. Then

$$\begin{aligned} \int \frac{x}{\sqrt{x^2-6x+13}}\,dx &= \int \frac{x}{\sqrt{(x-3)^2+4}}\,dx \\ &= \int \frac{3+2\tan(\theta)}{\sqrt{4\tan^2(\theta)+4}}\,2\sec^2(\theta)\,d\theta \\ &= \int \frac{3+2\tan(\theta)}{2\sec(\theta)} 2\sec^2(\theta)\,d\theta \\ &= \int 3\sec(\theta)+2\tan(\theta)\sec(\theta)\,d\theta \\ &= \int 3\sec(\theta)\,d\theta + \int 2\tan(\theta)\sec(\theta)\,d\theta \\ &= 3\ln|\sec(\theta)+\tan(\theta)| + 2\sec(\theta) + C. \end{aligned}$$

Now $\tan(\theta) = \frac{x-3}{2}$ implies $\sec(\theta) = \sqrt{1+\left(\frac{x-3}{2}\right)^2}$. So

$$\int \frac{x}{\sqrt{x^2-6x+13}}\,dx = 3\ln\left|\frac{x-3}{2} + \sqrt{1+\left(\frac{x-3}{2}\right)^2}\right| + 2\sqrt{1+\left(\frac{x-3}{2}\right)^2} + C.$$

(b) (10 points) Find the integral

$$\int \frac{2x+1}{x^3+2x^2+x}\,dx.$$

Answer:

We first factor the denominator: $x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2$. We have

$$\frac{2x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Solving for A, B, and C we get

$$A + B = 0 \qquad 2A + B + C = 2 \qquad A = 1$$

so that $B = -1$ and therefore $C = 1$.

Our integral is then

$$\begin{aligned} \int \frac{2x+1}{x(x+1)^2}\,dx &= \int \frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\,dx \\ &= \ln|x| - \ln|x+1| - \frac{1}{x+1} + C. \end{aligned}$$

2. (20 points)

(a) (10 points) Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx,$$

where $n \geq 2$ is an integer.

Answer:

Let $u = \sin^{n-1} x$ and $dv = \sin x\,dx$. Then $du = (n-1)\sin^{n-2} x \cos x\,dx$ and $v = -\cos x$. So integration by parts gives

$$\int \sin^n x\,dx = -\cos x \cdot \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx.$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \cdot \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.$$

We solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \cdot \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx.$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \cdot \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

(b) (10 points) Use your formula repeatedly to find

$$\int \sin^4 x\,dx$$

Answer:

If $n = 4$, then the reduction formula gives

$$\int \sin^4 x\,dx = -\frac{1}{4}\cos x \cdot \sin^3 x + \frac{3}{4}\int \sin^2 x\,dx.$$

If $n = 2$, then the reduction formula gives

$$\int \sin^2 x\,dx = -\frac{1}{2}\cos x \cdot \sin x + \frac{1}{2}\int\,dx.$$

Thus we have

$$\int \sin^4 x\,dx = -\frac{1}{4}\cos x \cdot \sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x \cdot \sin x + \frac{1}{2}\int\,dx\right)$$

$$= -\frac{1}{4}\cos x \cdot \sin^3 x - \frac{3}{8}\cos x \cdot \sin x + \frac{3}{8}x + C.$$

3. (20 points) If $a \neq 0$, evaluate

$$\int \sin^2(ax)\cos^2(ax)\,dx$$

in terms of a.

Answer:

Note that using double angle formulas, we have

$$\sin^2(ax)\cos^2(ax) = (\sin(ax)\cos(ax))^2 = \left(\frac{\sin(2ax)}{2}\right)^2$$
$$= \frac{\sin^2(2ax)}{4}$$

$$= \frac{1 - \cos(4ax)}{8}$$

Using the above identities, we obtain

$$\int \sin^2(ax)\cos^2(ax)\,dx = \int \frac{1-\cos(4ax)}{8}\,dx = \frac{x}{8} - \frac{\sin(4ax)}{32a} + C$$

4. (20 points) This a work problem with metric units. Assume that acceleration due to gravity is A meters per second per second. You should give your answer in joules as a multiple of $A\pi$. The density of water is a thousand kilograms per cubic meter.

Consider the region of the xy-plane bounded by the curve $y = x^2$ and the lines defined by $x = 0$ and $y = 3$. Rotate this region about the y-axis to obtain a solid region or bowl, which is filled with water. How much work is needed to pump the water about over the top of the bowl?

Answer:

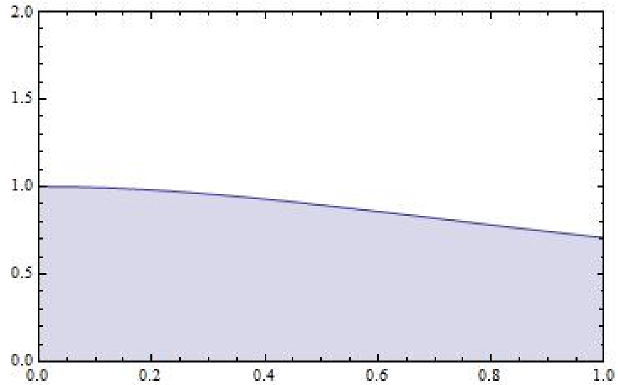
We need to divide the solid region into horizontal layers, one for each value of y. Thus it is convenient to write x as a function of y, namely $x = \sqrt{y}$ for $0 \le y \le 3$.

The radius of such a layer is x, so its area in square meters is $\pi x^2 = \pi y$, and its volume is therefore $\pi y\,dy$ cubic meters. This means its mass is $1000\pi y\,dy$ kilograms, so the gravitational force acting on it is $1000A\pi y\,dy$ newtons. The distance to the top of the solid is $(3 - y)$ meters, so the work needed to lift it is $1000A\pi y(3 - y)\,dy$ joules.

Thus the total amount of work in joules is

$$\begin{aligned}
\int_0^3 1000A\pi y(3-y)\,dy &= 1000A\pi \int_0^3 (3y - y^2)\,dy \\
&= 1000A\pi \left(\frac{3y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^3 \\
&= 1000A\pi \left(\frac{27}{2} - \frac{27}{3}\right) \\
&= 4500A\pi.
\end{aligned}$$

5. (20 points)



Consider the region $\mathcal{R}$ bounded by the x-axis, y-axis, the line $x = 1$, and $y = \frac{1}{\sqrt{1+x^2}}$.

(a) **(10 points)** Compute the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.

Answer:

We can use the disk method. The volume equals

$$\begin{aligned}\int_0^1 \pi \left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx =& \pi \int_0^1 \frac{1}{1+x^2} dx \\ =& \pi \left[\arctan(x)\right]_0^1 \\ =& \frac{\pi^2}{4}\end{aligned}$$

(b) **(10 points)** Compute the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.

Answer:

We can use the shell method. The volume equals

$$\int_0^1 2\pi x \frac{1}{\sqrt{1+x^2}} dx = \pi \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx$$

Now, we can let $u = 1 + x^2$ and $f(u) = \frac{1}{\sqrt{u}}$. So $du = 2xdx$.

$$\begin{aligned}\pi \int \frac{2x}{\sqrt{1+x^2}} dx =& \pi \int \frac{du}{\sqrt{u}} \\ =& 2\pi\sqrt{u} + C \\ =& 2\pi\sqrt{1+x^2} + C.\end{aligned}$$

So the definite integral is

$$\pi \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx = 2\pi [\sqrt{1+x^2}]_0^1$$
$$= 2\pi \left(\sqrt{2} - 1\right).$$

Scratch paper

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