Math 162: Calculus IIA

First Midterm Exam, Morning Edition ANSWERS

September 16, 2021

Trigonometric formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \dfrac{du}{u}$
- $\sec(\theta) = \dfrac{u^2 + 1}{2u}$
- $\tan(\theta) = \dfrac{u^2 - 1}{2u}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

1. (30 points) The problem has five parts, each worth 6 points. Each depends on having correct answers to the previous parts. You should not expect to get credit for an answer based on incorrect information.

Consider the integral

$$\int \frac{x^7\,dx}{x^4 + 10x^2 + 9}.$$

(a) Rewrite the integrand (NOT the integral, but the function being integrated) as the sum of a polynomial and a fraction in which the numerator is a polynomial of degree less that 4. SHOW YOUR WORK. You will not get credit for merely writing the correct answer.

Answer:

Here is the relevant long division of polynomials.

$$\begin{array}{r|l} & \qquad\qquad\quad x^3 - 10x \\ \hline x^4 + 10x^2 + 9 & x^7 \\ & -x^7 - 10x^5 \quad - 9x^3 \\ \hline & \quad - 10x^5 \quad - 9x^3 \\ & \quad\ 10x^5 + 100x^3 + 90x \\ \hline & \qquad\qquad 91x^3 + 90x \end{array}$$

We see that the quotient is $x^3 - 10x$, and the remainder is $91x^3 + 90x$. This means that the integrand is

$$x^3 - 10x + \frac{91x^3 + 90x}{x^4 + 10x^2 + 9}.$$

(b) Write the denominator of the fraction as a product of linear or quadratic factors.

Answer:

$$x^4 + 10x^2 + 9 = (x^2 + 1)(x^2 + 9).$$

(c) Are there any real values of x for which the integrand is not defined? If so, what are they and why?

Answer:

The denominator of the fraction is positive for all real values of x, so the integrand is always defined.

(d) Write the fraction you found in part (a) as a sum of fractions with linear functions as numerators, whose coefficients you may denote by letters of the alphabet. YOU NEED NOT FIND THE VALUES OF THESE CONSTANTS.

Answer:

$$\frac{91x^3+90x}{x^4-10x^2+9} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+9}.$$

(e) Write the original integral in terms of the constants you named in part (d). You may use the fact that

$$\int \frac{x\,dx}{x^2+k^2} = \frac{1}{2}\ln|x^2+k^2| + c \quad \text{and} \quad \int \frac{dx}{x^2+k^2} = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + c.$$

Answer:

$$\begin{aligned}
\int \frac{x^7\,dx}{x^4+10x^2+9} &= \int (x^3-10x)\,dx + \int \frac{(91x^3+90x)\,dx}{x^4+10x^2+9} \\
&= \int (x^3-10x)\,dx + A\int \frac{x\,dx}{x^2+1} + B\int \frac{dx}{x^2+1} \\
&\quad + C\int \frac{x\,dx}{x^2+9} + D\int \frac{dx}{x^2+9} \\
&= \frac{x^4}{4} - 5x + \left(\frac{A}{2}\right)\ln|x^2+1| + B\arctan x \\
&\quad + \left(\frac{C}{2}\right)\ln|x^2+9| + \left(\frac{D}{3}\right)\arctan\left(\frac{x}{3}\right) + c.
\end{aligned}$$

2. (20 points) This a work problem with metric units. Assume that acceleration due to gravity is A meters per second per second. You should give your answer in joules as a multiple of $A\pi$. The density of water is a thousand kilograms per cubic meter.

Consider the region of the xy-plane bounded by the curve $y = 3 - x^2$ and the lines defined by $x = 0$ and $y = 0$. Rotate this region about the y-axis to obtain a solid region or tank, which is filled with water. How much work is needed to pump the water about out the top of the tank?

Answer:

We need to divide the solid region into horizontal layers, one for each value of y. Thus it is convenient to write x as a function of y, namely $x = \sqrt{3-y}$ for $0 \le y \le 3$.

The radius of such a layer is x, so its area in square meters is $\pi x^2 = \pi(3-y)$, and its volume is therefore $\pi(3-y)\,dy$ cubic meters. This means its mass is $1000\pi(3-y)\,dy$ kilograms, so the gravitational force acting on it is $1000A\pi(3-y)\,dy$ newtons. The distance to the top of the solid is $(3-y)$ meters, so the work needed to lift it is $1000A\pi(3-y)(3-y)\,dy$ joules.

Thus the total amount of work in joules is

$$\begin{aligned}\int_0^3 1000A\pi(3-y)(3-y)\,dy &= 1000A\pi\int_0^3 (9-6y+y^2)\,dy\\ &= 1000A\pi\left(9y-3y^2+\frac{y^3}{3}\right)\Bigg|_0^3\\ &= 1000A\pi\left(27-27+\frac{27}{3}\right)\\ &= 9000A\pi.\end{aligned}$$

3. (30 points)

(a) (15 points) Use integration by parts twice to find a formula for

$$\int x^{n+1}e^{x^2}\,dx \qquad \text{in terms of} \qquad \int x^{n-3}e^{x^2}\,dx \qquad \text{for } n>3.$$

Hint: start with $x^{n+1}e^{x^2} = \left(\frac{1}{2}x^n\right)\left(2xe^{x^2}\right)$, then use

$$u = \frac{1}{2}x^n, \quad dv = 2xe^{x^2}\,dx.$$

Answer:

Using integration by parts twice, we have

$$\begin{aligned}\int x^{n+1}e^{x^2}\,dx &= \int \left(\frac{1}{2}x^n\right)\left(2xe^{x^2}\right)\,dx\\ &= \frac{1}{2}\int x^n\left(e^{x^2}\right)'\,dx\\ &= \frac{1}{2}\left(x^ne^{x^2} - \int nx^{n-1}e^{x^2}\,dx\right)\\ &= \frac{1}{2}x^ne^{x^2} - \frac{n}{2}\int x^{n-1}e^{x^2}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}x^n e^{x^2} - \frac{n}{2}\int \left(\frac{1}{2}x^{n-2}\right)\left(e^{x^2}\right)' \, dx \\
&= \frac{1}{2}x^n e^{x^2} - \frac{n}{4}\left(x^{n-2}e^{x^2} - \int (n-2)x^{n-3}e^{x^2}\, dx\right) \\
&= \frac{1}{2}x^n e^{x^2} - \frac{n}{4}x^{n-2}e^{x^2} + \frac{n(n-2)}{4}\int x^{n-3}e^{x^2}\, dx
\end{aligned}
$$

(b) (15 points) Let

$$I_n = \int x^{n+1}e^{x^2}\, dx.$$

Use your formula of part (a) to find I_8. You should not expect to get credit for an answer based on an incorrect formula.

Answer:

$$
\begin{aligned}
I_8 &= \int x^9 e^{x^2}\, dx \\
&= \frac{1}{2}x^8 e^{x^2} - \frac{8}{4}x^6 e^{x^2} + \frac{8\cdot 6}{4}\int x^5 e^{x^2}\, dx \\
&= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 12\left(\frac{1}{2}x^4 e^{x^2} - \frac{4}{4}x^2 e^{x^2} + \frac{4\cdot 2}{4}\int x e^{x^2}\, dx\right) \\
&= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 6x^4 e^{x^2} - 12x^2 e^{x^2} + 24\int x e^{x^2}\, dx \\
&= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 6x^4 e^{x^2} - 12x^2 e^{x^2} + 12e^{x^2} + C
\end{aligned}
$$

4. (20 points)

Compute the following integral:

$$\int (1-x^2)^{3/2} dx.$$

Answer:

Make the substitution $x = \cos(\theta)$. $\sqrt{1-x^2} = \sin(\theta)$ and $dx = -\sin(\theta)d\theta$. Now the integral is

$$-\int \sin^4(\theta)d\theta.$$

Use the trig identity $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$. Now the integral is

$$-\int \left(\frac{1-\cos(2\theta)}{2}\right)^2 d\theta = -\frac{1}{4}\int 1 - 2\cos(2\theta) + \cos^2(2\theta)d\theta.$$

Use the trig identity $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$. Now the integral is

$$-\frac{1}{4}\int 1 - 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}d\theta = -\frac{1}{4}\int 3/2 - 2\cos(2\theta) + \frac{\cos(4\theta)}{2}d\theta.$$

Integrate to get

$$-\frac{1}{4}\left[3\theta/2 - \sin(2\theta) + \frac{\sin(4\theta)}{8}\right] + C.$$

Use the trig identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to get

$$-\frac{1}{4}\left[3\theta/2 - 2\sin(\theta)\cos(\theta) + \frac{\sin(2\theta)\cos(2\theta)}{4}\right] + C.$$

Apply this identity again, along with $\cos(2\theta) = 2\cos^2(\theta) - 1$ to get

$$-\frac{1}{4}\left[3\theta/2 - 2\sin(\theta)\cos(\theta) + \frac{\sin(\theta)\cos(\theta)(\cos^2(\theta)-1)}{2}\right] + C.$$

Finally, substitute back $x = \cos(\theta)$ and $\sqrt{1-x^2} = \sin(\theta)$ to get

$$-\frac{1}{4}\left[\frac{3}{2}\arccos(x) - 2x\sqrt{1-x^2} + \frac{x\sqrt{1-x^2}(2x^2-1)}{2}\right] + C.$$