

Math 162: Calculus IIA

First Midterm Exam, Morning Edition ANSWERS

October 9, 2020

Trigonometric formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sin(2x) = 2 \sin(x) \cos(x)$
- $\cos^2(x) = \frac{1 + \cos(2x)}{2}$
- $\sin^2(x) = \frac{1 - \cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \frac{du}{u}$
- $\sec(\theta) = \frac{u^2 + 1}{2u}$
- $\tan(\theta) = \frac{u^2 - 1}{2u}$

Integration by parts:

$$\int u dv = uv - \int v du$$

1. (30 points) The problem has five parts, each worth 6 points. Each depends on having correct answers to the previous parts. You should not expect to get credit for an answer based on incorrect information.

Consider the integral

$$\int \frac{x^7 dx}{x^4 + 10x^2 + 9}.$$

- (a) Rewrite the integrand (NOT the integral, but the function being integrated) as the sum of a polynomial and a fraction in which the numerator is a polynomial of degree less than 4. SHOW YOUR WORK. You will not get credit for merely writing the correct answer.

Answer:

Here is the relevant long division of polynomials.

$$\begin{array}{r}
 \\
 x^7 \\
 \underline{- x^7 - 10x^5} \\
 - 10x^5 \\
 \underline{- 9x^3} \\
 10x^5 + 100x^3 + 90x \\
 \underline{91x^3 + 90x}
 \end{array}$$

We see that the quotient is $x^3 - 10x$, and the remainder is $91x^3 + 90x$. This means that the integrand is

$$x^3 - 10x + \frac{91x^3 + 90x}{x^4 + 10x^2 + 9}.$$

- (b) Write the denominator of the fraction as a product of linear or quadratic factors.

Answer:

$$x^4 + 10x^2 + 9 = (x^2 + 1)(x^2 + 9).$$

- (c) Are there any real values of x for which the integrand is not defined? If so, what are they and why?

Answer:

The denominator of the fraction is positive for all real values of x , so the integrand is always defined.

- (d) Write the fraction you found in part (a) as a sum of fractions with linear functions as numerators, whose coefficients you may denote by letters of the alphabet. YOU NEED NOT FIND THE VALUES OF THESE CONSTANTS.

Answer:

$$\frac{91x^3 + 90x}{x^4 - 10x^2 + 9} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 9}.$$

- (e) Write the original integral in terms of the constants you named in part (d). You may use the fact that

$$\int \frac{x \, dx}{x^2 + k^2} = \frac{1}{2} \ln |x^2 + k^2| + c \quad \text{and} \quad \int \frac{dx}{x^2 + k^2} = \frac{1}{k} \arctan \left(\frac{x}{k} \right) + c.$$

Answer:

$$\begin{aligned} \int \frac{x^7 \, dx}{x^4 + 10x^2 + 9} &= \int (x^3 - 10x) \, dx + \int \frac{(91x^3 + 90x) \, dx}{x^4 + 10x^2 + 9} \\ &= \int (x^3 - 10x) \, dx + A \int \frac{x \, dx}{x^2 + 1} + B \int \frac{dx}{x^2 + 1} \\ &\quad + C \int \frac{x \, dx}{x^2 + 9} + D \int \frac{dx}{x^2 + 9} \\ &= \frac{x^4}{4} - 5x + \left(\frac{A}{2} \right) \ln |x^2 + 1| + B \arctan x \\ &\quad + \left(\frac{C}{2} \right) \ln |x^2 + 9| + \left(\frac{D}{3} \right) \arctan \left(\frac{x}{3} \right) + c. \end{aligned}$$

2. (20 points) This a work problem with metric units. Assume that acceleration due to gravity is A meters per second per second. You should give your answer in joules as a multiple of $A\pi$. The density of water is a thousand kilograms per cubic meter.

Consider the region of the xy -plane bounded by the curve $y = 3 - x^2$ and the lines defined by $x = 0$ and $y = 0$. Rotate this region about the y -axis to obtain a solid region or tank, which is filled with water. How much work is needed to pump the water about out the top of the tank?

Answer:

We need to divide the solid region into horizontal layers, one for each value of y . Thus it is convenient to write x as a function of y , namely $x = \sqrt{3 - y}$ for $0 \leq y \leq 3$.

The radius of such a layer is x , so its area in square meters is $\pi x^2 = \pi(3 - y)$, and its volume is therefore $\pi(3 - y) dy$ cubic meters. This means its mass is $1000\pi(3 - y) dy$ kilograms, so the gravitational force acting on it is $1000A\pi(3 - y) dy$ newtons. The distance to the top of the solid is $(3 - y)$ meters, so the work needed to lift it is $1000A\pi(3 - y)(3 - y) dy$ joules.

Thus the total amount of work in joules is

$$\begin{aligned} \int_0^3 1000A\pi(3 - y)(3 - y) dy &= 1000A\pi \int_0^3 (9 - 6y + y^2) dy \\ &= 1000A\pi \left(9y - 3y^2 + \frac{y^3}{3} \right) \Big|_0^3 \\ &= 1000A\pi \left(27 - 27 + \frac{27}{3} \right) \\ &= 9000A\pi. \end{aligned}$$

3. (30 points)

(a) (15 points) Use integration by parts twice to find a formula for

$$\int x^{n+1} e^{x^2} dx \quad \text{in terms of} \quad \int x^{n-3} e^{x^2} dx \quad \text{for } n > 3.$$

Hint: start with $x^{n+1} e^{x^2} = \left(\frac{1}{2}x^n\right) (2xe^{x^2})$, then use

$$u = \frac{1}{2}x^n, \quad dv = 2xe^{x^2} dx.$$

Answer:

Using integration by parts twice, we have

$$\begin{aligned} \int x^{n+1} e^{x^2} dx &= \int \left(\frac{1}{2}x^n\right) (2xe^{x^2}) dx \\ &= \frac{1}{2} \int x^n (e^{x^2})' dx \\ &= \frac{1}{2} \left(x^n e^{x^2} - \int nx^{n-1} e^{x^2} dx \right) \\ &= \frac{1}{2} x^n e^{x^2} - \frac{n}{2} \int x^{n-1} e^{x^2} dx \\ &= \frac{1}{2} x^n e^{x^2} - \frac{n}{2} \int \left(\frac{1}{2}x^{n-2}\right) (e^{x^2})' dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}x^n e^{x^2} - \frac{n}{4} \left(x^{n-2} e^{x^2} - \int (n-2)x^{n-3} e^{x^2} dx \right) \\
 &= \frac{1}{2}x^n e^{x^2} - \frac{n}{4}x^{n-2} e^{x^2} + \frac{n(n-2)}{4} \int x^{n-3} e^{x^2} dx
 \end{aligned}$$

(b) (15 points) Let

$$I_n = \int x^{n+1} e^{x^2} dx.$$

Use your formula of part (a) to find I_8 . You should not expect to get credit for an answer based on an incorrect formula.

Answer:

$$\begin{aligned}
 I_8 &= \int x^9 e^{x^2} dx \\
 &= \frac{1}{2}x^8 e^{x^2} - \frac{8}{4}x^6 e^{x^2} + \frac{8 \cdot 6}{4} \int x^5 e^{x^2} dx \\
 &= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 12 \left(\frac{1}{2}x^4 e^{x^2} - \frac{4}{4}x^2 e^{x^2} + \frac{4 \cdot 2}{4} \int x e^{x^2} dx \right) \\
 &= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 6x^4 e^{x^2} - 12x^2 e^{x^2} + 24 \int x e^{x^2} dx \\
 &= \frac{1}{2}x^8 e^{x^2} - 2x^6 e^{x^2} + 6x^4 e^{x^2} - 12x^2 e^{x^2} + 12e^{x^2} + C
 \end{aligned}$$

4. (20 points)

Compute the following integral:

$$\int (1-x^2)^{3/2} dx.$$

Answer:

Make the substitution $x = \cos(\theta)$. $\sqrt{1-x^2} = \sin(\theta)$ and $dx = -\sin(\theta)d\theta$. Now the integral is

$$-\int \sin^4(\theta)d\theta.$$

Use the trig identity $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$. Now the integral is

$$-\int \left(\frac{1-\cos(2\theta)}{2} \right)^2 d\theta = -\frac{1}{4} \int 1 - 2\cos(2\theta) + \cos^2(2\theta) d\theta.$$

Use the trig identity $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$. Now the integral is

$$-\frac{1}{4} \int 1 - 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2} d\theta = -\frac{1}{4} \int 3/2 - 2\cos(2\theta) + \frac{\cos(4\theta)}{2} d\theta.$$

Integrate to get

$$-\frac{1}{4} \left[3\theta/2 - \sin(2\theta) + \frac{\sin(4\theta)}{8} \right] + C.$$

Use the trig identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to get

$$-\frac{1}{4} \left[3\theta/2 - 2\sin(\theta)\cos(\theta) + \frac{\sin(2\theta)\cos(2\theta)}{4} \right] + C.$$

Apply this identity again, along with $\cos(2\theta) = 2\cos^2(\theta) - 1$ to get

$$-\frac{1}{4} \left[3\theta/2 - 2\sin(\theta)\cos(\theta) + \frac{\sin(\theta)\cos(\theta)(\cos^2(\theta) - 1)}{2} \right] + C.$$

Finally, substitute back $x = \cos(\theta)$ and $\sqrt{1-x^2} = \sin(\theta)$ to get

$$-\frac{1}{4} \left[\frac{3}{2} \arccos(x) - 2x\sqrt{1-x^2} + \frac{x\sqrt{1-x^2}(2x^2-1)}{2} \right] + C.$$