Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 11, 2016

1. (20 points)

(a) Use integration by parts twice to express $I_n = \int x^n \cos x \, dx$ in terms of I_{n-2} for $n \geq 2$.

Answer:

Apply integration by parts, with $u = x^n$ and $dv = \cos x \, dx$ and therefore $du = nx^{n-1}$ and $v = \sin x$, to obtain:

$$I_n = x^n \sin x - n \int x^{n-1} \sin x \, dx + C$$

Apply integration by parts again, this time with $u = x^{n-1}$ and $dv = \sin x \, dx$ and therefore $du = (n-1)x^{n-2}$ and $v = -\cos x$, to obtain:

$$\begin{aligned} I_n &= x^n \sin x - n \left[-x^{n-1} \cos x - \int (-\cos x)(n-1)x^{n-2} \, dx \right] \\ &= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx \\ &= x^n \sin x + nx^{n-1} \cos x - n(n-1) I_{n-2} \end{aligned}$$

(b) Use the formula of part (a) repeatedly to find I_4.

You will not get partial credit here if the formula you are using is incorrect.

Answer:

First note that:

$$I_0 = \int \cos x \, dx = \sin x + C$$

By the formula of part (a) then:

$$\begin{aligned} I_2 &= x^2 \sin x + 2x \cos x - 2I_0 \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

By the formula of part (a) then:

$$\begin{aligned} I_4 &= x^4 \sin x + 4x^3 \cos x - 12 I_2 \\ &= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + C. \end{aligned}$$

2. (20 points)

A heavy rope, 20 ft long, weighs 0.5 lbs/ft and hangs over the edge of a building 100 ft high. How much work is done in pulling half the rope to the top of the building?

Answer:

Solution:

The bottom half of the rope is pull up 10 feet, and finding the work requires no caluclus:

work for bottom half = (10 ft of rope) * (0.5 lbs per foot) * (lifting 10 feet) = 50 ft-lbs.

The top half of the rope requires work $= \int_0^{10} (0.5)x \, dx = 25$ ft-lbs. Thus, total work is 75 ft-lbs.

3. (20 points) Consider the region bounded by the x-axis and the curve $y = \sin x$ for $0 \le x \le \pi$.

(a) Find the volume of the solid obtained by rotating it about the x-axis.

Answer:

(b) Find the volume of the solid obtained by rotating the same region about the y-axis.

Answer:

Solution:

a) Integrating with respect to x makes this a washer method problem with

$$\begin{aligned} V &= \int_0^\pi \pi \sin^2 x\,dx \\ &= \pi \int_0^\pi \left(\frac{1-\cos 2x}{2}\right) dx \qquad \text{by the half angle formula} \\ &= \pi \left(\int_0^\pi \frac{dx}{2} - \int_0^{2\pi} \frac{\cos u}{4}\,du\right) \qquad \text{where } u = 2x \\ &= \left(\frac{\pi x}{2}\right)\Bigg|_0^\pi - \left(\frac{\sin u}{4}\right)\Bigg|_0^{2\pi} \\ &= \frac{\pi^2}{2}. \end{aligned}$$

b) Integrating with respect to x makes this a shell method problem with

$$V = \int_0^\pi 2\pi x \sin x\,dx = 2\pi \int_0^\pi x \sin x\,dx$$

For this we need integration by parts with

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

Then we have

$$\begin{aligned} V &= 2\pi \int_{x=0}^{x=\pi} u\,dv \\ &= 2\pi \left(uv\Bigg|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v\,du \right) \\ &= 2\pi \left(-x\cos x\Bigg|_0^\pi + \int_0^\pi \cos x\,dx \right) \\ &= 2\pi\,(-x\cos x + \sin x)\Bigg|_0^\pi \\ &= 2\pi(\pi - 0) = 2\pi^2. \end{aligned}$$

4. (20 points)

Compute the definite integral

$$\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{(2x^2-3)^{3/2}}.$$

Answer:

Solution:

Let $u = \sqrt{2}x$. Then $du = \sqrt{2}\,dx$. Also when $x = \sqrt{2}$, $u = 2$, and when $x = \sqrt{6}$, $u = 2\sqrt{3}$. Thus the definite integral becomes

$$\frac{\sqrt{2}}{2}\int_2^{2\sqrt{3}} \frac{du}{(u^2-3)^{3/2}}.$$

Now let $u = \sqrt{3}\sec\theta$. Then $du = \sqrt{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-3} = \sqrt{3}\tan\theta$. Also when $u = 2$, $\sec\theta = 2/\sqrt{3}$ so that $\theta = \pi/6$, and when $u = 2\sqrt{3}$, $\sec\theta = 2$ so that $\theta = \pi/3$. Then the definite integral becomes

$$\begin{aligned}
\frac{\sqrt{2}}{2}\int_{\pi/6}^{\pi/3} \frac{\sqrt{3}\sec\theta\tan\theta\,d\theta}{3\sqrt{3}\tan^3\theta} &= \frac{\sqrt{2}}{6}\int_{\pi/6}^{\pi/3} \frac{\sec\theta\,d\theta}{\tan^2\theta} \\
&= \frac{\sqrt{2}}{6}\int_{\pi/6}^{\pi/3} \frac{\cos\theta\,d\theta}{\sin^2\theta} \\
&= -\frac{\sqrt{2}}{6}\left.\frac{1}{\sin\theta}\right|_{\pi/6}^{\pi/3} \\
&= \frac{\sqrt{2}}{3}\left(1-\frac{1}{\sqrt{3}}\right) \quad \text{or} \quad \frac{\sqrt{2}}{3}-\frac{\sqrt{6}}{9}.
\end{aligned}$$

5. (20 points)

Evaluate the following integral:

$$\int \frac{3x}{(x+1)(x^3+1)}dx.$$

Answer:

Solution: By partial fractions we have

$$\begin{aligned}
\frac{3x}{(x+1)(x^3+1)} &= \frac{3x}{(x+1)^2(x^2-x+1)} \\
&= \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2-x+1} \qquad (1)\\
&= \frac{A(x+1)(x^2-x+1)+B(x^2-x+1)+(Cx+D)(x+1)^2}{(x+1)^2(x^2-x+1)} \\
&= \frac{(A+C)x^3+(B+2C+D)x^2+(-B+C+2D)x+(A+B+D)}{(x+1)^2(x^2-x+1)}
\end{aligned}$$

By comparing numerators we must have $A+C=0$, $B+2C+D=0$, $-B+C+2D=3$ and $A+B+D=0$. From this we get $A=C=0$, $B=-1$ and $D=1$.

Alternatively, we can use Heaviside's method to find the constants. Multiply both sides of (1) by $(x+1)^2$ and get

$$\frac{3x}{x^2-x+1} = A(x+1)+B+(x+1)^2\frac{Cx+D}{x^2-x+1}$$

Setting $x=-1$ gives

$$B=\frac{-3}{3}=-1.$$

Subtarcting the B term from both sides of (1) gives

$$\begin{aligned}\frac{A}{x+1}+\frac{Cx+D}{x^2-x+1} &= \frac{3x}{(x+1)(x^3+1)}+\frac{1}{(x+1)^2}\\ &= \frac{3x+x^2-x+1}{(x+1)^2(x^2-x+1)}\\ &= \frac{x^2+2x+1}{(x+1)^2(x^2-x+1)}\\ &= \frac{1}{x^2-x+1}.\end{aligned}$$

From this we see that $A=0$, $C=0$ and $D=1$ as before.

Thus one gets

$$\int \frac{3x}{(x+1)(x^3+1)}dx = -\int\frac{dx}{(x+1)^2}+\int\frac{dx}{x^2-x+1}.$$

The first integral is done by substitution $u=x+1$. For the second integral, we observe that

$$\frac{1}{x^2-x+1}=\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,.$$

Therefore we use substitution $u=2/\sqrt{3}(x-1/2)$ and $du=2/\sqrt{3}dx$, then the second integral is

$$\int\frac{dx}{x^2-x+1}=\frac{2}{\sqrt{3}}\int\frac{du}{u^2+1}=\frac{2}{\sqrt{3}}\tan^{-1}u=\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x-\frac{1}{2}))\,.$$

Thus we have

$$\int \frac{3x}{(x+1)(x^3+1)}dx=\frac{1}{x+1}+\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x-\frac{1}{2}))+K\,.$$

(b)We will use the asubstitution $u = \sec x$, which gives

$$\begin{aligned} du &= \sec x \tan x dx \\ \tan^2 x &= \sec^2 x - 1 = u^2 - 1 \\ \sec 0 &= 1 \\ \sec \pi/4 &= \sqrt{2}, \end{aligned}$$

so the integral is

$$\int_1^{\sqrt{2}} (u^2 - 1) du = \left(\frac{u^3}{3} - u \right) \Big|_1^{\sqrt{2}} \qquad =$$