Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 10, 2015

1. (20 points)

(a) Use integration by parts to express $I_n = \int_0^{\pi/2} \cos^n x\,dx$ in terms of I_{n-2} for $n \geq 2$.

Solution: a.) Let $u = cos^{n-1}x$ and $dv = \cos x\,dx$, so $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. Use $\sin^2 x = 1 - \cos^2 x$ to get

$$\begin{aligned} I_n = \int_0^{\pi/2} \cos^n x\,dx &= uv\Big|_0^{\pi/2} - \int_{x=0}^{x=\pi/2} v\,du \\ &= cos^{n-1}x \sin x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x\,dx \\ &= (n-1)\int_0^{\pi/2} \cos^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \cos^n x\,dx. \end{aligned}$$

We have

$$n\int_0^{\pi/2} \cos^n x\,dx = (n-1)\int_0^{\pi/2} \cos^{n-2} x\,dx,$$

and therefore

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(b) Use the formula of part (a) repeatedly to find I_6.

You will not get partial credit here if the formula you are using is incorrect.

Solution: b.) Note that

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

We have

$$\begin{aligned} I_6 &= \frac{5}{6} I_4 \\ &= \frac{5 \cdot 3}{6 \cdot 4} I_2 \\ &= \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} I_0 \\ &= \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}. \end{aligned}$$

2. (20 points)

A tank has the shape of an inverted pyramid (point down) with a square base. The tank is 8 ft tall and the area of its base is 16 ft^2. The tank is filled with fluid to a height of 2 ft. Calculate the work done in pumping all the fluid from the tank. Assume that the density of the fluid is 12 pounds per cubic foot.

Solution:

First, note that the lenght of a side of the base is $\sqrt{16} = 4$ ft. Let y be the height of the fluid from the bottom of the tank. Then, the length of a side of the cross-sectional square is $\frac{4}{8}y = \frac{1}{2}y$. We can compute the volume of the thin layer of the fluid at height y as

$$dV = \left(\frac{1}{2}y\right)^2 dy = \frac{1}{4}y^2\, dy.$$

The work then can be computed by multiplying it by the weight density 12 and the height it travels $8 - y$ and integrating from 0 to 2:

$$\begin{aligned} W &= \int_0^2 (8-y)\, 12\, \frac{1}{4} y^2\, dy \\ &= \int_0^2 24y^2 - 3y^3\, dy \\ &= \left[8y^3 - \frac{3}{4}y^4\right]_0^2 = 64 - 12 = 52 \text{ ft-lb}. \end{aligned}$$

3. (20 points) Hole in the sphere problem. A hole of radius r is bored through the center of a sphere of radius $R > r$. We want to find the volume of the remaining portion of the sphere.

(a) Set up the integral as a solid of revolution around the y-axis using the shell method. Evaluate the integral

(b) Set up the integral using the washer method. Do not evaluate.

Solution: By symmetry, the sphere with a hole can be obtained by rotating a disk with a middle removed around the x or y-axis. To rotate around y-axis, we remove a vertical strip $-r \leq x \leq r$ and rotate the remaining region around the y-axis. By symmetry the volume of the region sitting above the x-axis is half of the full volume. From the equation of the circle $x^2 + y^2 = R^2$.

(a) Using the shell method

$$V = 2\int_r^R 2\pi xy dx = 4\pi \int_r^R x\sqrt{R^2 - x^2} dx$$

To evaluate the integral, do a u-substitution $u = R^2 - x^2$, $du = -2x$. Thus

$$\begin{aligned} V &= -2\pi \int_{R^2-r^2}^{0} u^{\frac{1}{2}} du \\ &= -\frac{4\pi}{3} u^{\frac{3}{2}} \Big|_{R^2-r^2}^{0} \\ &= \frac{4\pi}{3}(R^2 - r^2)^{\frac{3}{2}} \end{aligned}$$

(b) Using the washer method, we have to integrate along the y-axis from $y = -\sqrt{R^2 - r^2}$ to $\sqrt{R^2 - r^2}$

$$\begin{aligned} V &= \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\pi x^2 - \pi r^2\right) dy \\ &= \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\pi(R^2 - y^2) - \pi r^2\right) dy \\ &= 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - r^2 - y^2\right) dy, \text{ by symmetry} \end{aligned}$$

4. (20 points)

(a) Let $a > 0$ be a fixed positive number. Compute the definite integral

$$\int_{a\sqrt{2}}^{2a} \frac{dx}{\sqrt{x^2 - a^2}}.$$

(b) Find the integral

$$\int \frac{1}{\sqrt{x^2 + 6x + 10}} dx.$$

Solution: (a)

We set $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$. Also when $x = a\sqrt{2}$, $\sec\theta = \sqrt{2}$ so that $\theta = \pi/4$, and when $x = 2a$, $\sec\theta = 2$ so that $\theta = \pi/3$. The definite integral becomes

$$\int_{a\sqrt{2}}^{2a} \frac{dx}{\sqrt{x^2 - a^2}} = \int_{\pi/4}^{\pi/3} \sec\theta \, d\theta.$$

Now let $u = \sec\theta + \tan\theta$ so $\sec\theta d\theta = du/u$. When $\theta = \pi/4$,$u = 1 + \sqrt{2}$ and when $\theta = \pi/3$, $u = 2 + \sqrt{3}$, so the definite integral becomes

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \sec\theta \, d\theta &= \int_{1+\sqrt{2}}^{2+\sqrt{3}} \frac{du}{u} \\
&= \ln u\big|_{1+\sqrt{2}}^{2+\sqrt{3}} \\
&= \ln(2+\sqrt{3}) - \ln(1+\sqrt{2}) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) \\
&= \ln\left(\frac{(2+\sqrt{3})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right) = \ln\left(\frac{(\sqrt{3}+2)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right) \\
&= \ln\left(\sqrt{6} + 2\sqrt{2} - \sqrt{3} - 2\right)
\end{aligned}$$

(b) We complete the square $x^2 + 6x + 10 = (x+3)^2 + 1$. Then consider the substitution $u = x + 3$, so that $du = dx$, and we find

$$\int \frac{1}{\sqrt{x^2 + 6x + 10}} \, dx = \int \frac{1}{\sqrt{(x+3)^2 + 1}} \, dx = \int \frac{1}{\sqrt{u^2 + 1}} \, du.$$

Next we use a trig substitution. Let $u = \tan\theta$. Then $du = \sec^2\theta d\theta$ and $\sqrt{u^2 + 1} = 1\sec\theta$, so that

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 + 6x + 10}} \, dx = \int \frac{1}{\sqrt{u^2 + 1}} \, du = \int \frac{1}{\sec\theta} \sec^2\theta \, d\theta &= \int \sec\theta \, d\theta \\
= \ln|\sec\theta + \tan\theta| + C &= \ln|\sqrt{u^2 + 1} + u| + C \\
= \ln|\sqrt{(x+3)^2 + 1} + x + 3| + C & \\
= \ln|x + 3 + \sqrt{x^2 + 6x + 10}| + C &
\end{aligned}$$

5. (20 points)

(a) Find the integral

$$\int \sin^4 x \cos^3 x \, dx.$$

(b) Evaluate the integral

$$\int \frac{x^2+x+2}{x^3+x^2+x+1}\,dx.$$

Solution: (a)

Note that $\cos^3 x = \cos^2 x \cdot \cos x = (1-\sin^2 x)\cos x$. So, letting $u = \sin x$, we have $du = \cos x dx$ and

$$\begin{aligned}
\int \sin^4 x \cos^3 x\,dx &= \int \sin^4 x(1-\sin^2 x)\cos x\,dx \\
&= \int u^4(1-u^2)du \\
&= \int u^4 - u^6 du \\
&= \frac{u^5}{5} - \frac{u^7}{7} + C \\
&= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C
\end{aligned}$$

(b) Factoring the denominator,$x^3+x^2+x+1 = (x+1)(x^2+1)$ means we may write

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Multiplying both sides by $x+1$ gives

$$\frac{x^2+x+2}{(x^2+1)} = A + (x+1)\left(\frac{Bx+C}{x^2+1}\right),$$

so setting $x \mapsto -1$ gives $A = 1$. Hence we get

$$\begin{aligned}
\frac{Bx+C}{x^2+1} &= \frac{x^2+x+2}{(x+1)(x^2+1)} - \frac{A}{x+1} = \frac{x^2+x+2}{(x+1)(x^2+1)} - \frac{1}{x+1} \\
&= \frac{x^2+x+2-(x^2+1)}{(x+1)(x^2+1)} = \frac{x+1}{(x+1)(x^2+1)} = \frac{1}{x^2+1},
\end{aligned}$$

so $B = 0$ and $C = 1$.

Thus our integral is

$$\begin{aligned}
\int \frac{x^2+x+2}{x^3+x^2+x+1}\,dx &= \int \left(\frac{1}{x+1} + \frac{1}{x^2+1}\right)\,dx \\
&= \ln|x+1| + \arctan x + c.
\end{aligned}$$