Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 18, 2011

1. (20 points) Evaluate the following integrals:
(a) (10 points)

$$\int \frac{3x}{(x+1)(x^3+1)} dx.$$

(b) (10 points)

$$\int_0^{\pi/2} \sin^4 x dx.$$

Solution: (a) By partial fractions we have

$$\begin{aligned} \frac{3x}{(x+1)(x^3+1)} &= \frac{3x}{(x+1)^2(x^2-x+1)} \\ &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} \qquad (1) \\ &= \frac{A(x+1)(x^2-x+1) + B(x^2-x+1) + (Cx+D)(x+1)^2}{(x+1)^2(x^2-x+1)} \\ &= \frac{(A+C)x^3 + (B+2C+D)x^2 + (-B+C+2D)x + (A+B+D)}{(x+1)^2(x^2-x+1)} \end{aligned}$$

By comparing numerators we must have $A+C=0$, $B+2C+D=0$, $-B+C+2D=3$ and $A+B+D=0$. From this we get $A=C=0$, $B=-1$ and $D=1$.

Alternatively, we can use Heaviside's method to find the constants. Multiply both sides of (1) by $(x+1)^2$ and get

$$\frac{3x}{x^2-x+1} = A(x+1) + B + (x+1)^2 \frac{Cx+D}{x^2-x+1}$$

Setting $x=-1$ gives

$$B = \frac{-3}{3} = -1.$$

Subtracting the B term from both sides of (1) gives

$$\begin{aligned}\frac{A}{x+1}+\frac{Cx+D}{x^2-x+1} &= \frac{3x}{(x+1)(x^3+1)}+\frac{1}{(x+1)^2}\\ &= \frac{3x+x^2-x+1}{(x+1)^2(x^2-x+1)}\\ &= \frac{x^2+2x+1}{(x+1)^2(x^2-x+1)}\\ &= \frac{1}{x^2-x+1}.\end{aligned}$$

From this we see that $A=0$, $C=0$ and $D=1$ as before.

Thus one gets

$$\int \frac{3x}{(x+1)(x^3+1)}dx = -\int \frac{dx}{(x+1)^2}+\int \frac{dx}{x^2-x+1}.$$

The first integral is done by substitution $u=x+1$. For the second integral, we observe that

$$\frac{1}{x^2-x+1}=\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,.$$

Therefore we use substitution $u=2/\sqrt{3}(x-1/2)$ and $du=2/\sqrt{3}dx$, then the second integral is

$$\int \frac{dx}{x^2-x+1}=\frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}=\frac{2}{\sqrt{3}}\tan^{-1}u=\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x-\frac{1}{2}))\,.$$

Thus we have

$$\int \frac{3x}{(x+1)(x^3+1)}dx=\frac{1}{x+1}+\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x-\frac{1}{2}))+K\,.$$

(b) We will use the double angle formulas $\sin^2\theta=(1-\cos 2\theta)/2$ and $\cos^2\theta=(1+\cos 2\theta)/2$. We have

$$\begin{aligned}\int_0^{\pi/2}\sin^4 x\,dx &= \int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)^2 dx\\ &= \frac{1}{4}\int_0^{\pi/2}\left(1-2\cos 2x+\cos^2 2x\right)\,dx\\ &= \frac{1}{4}\int_0^{\pi/2}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)\,dx\\ &= \frac{1}{8}\int_0^{\pi/2}\left(3-4\cos 2x+\cos 4x\right)\,dx\\ &= \frac{1}{8}\left(3x-2\sin 2x+\frac{\sin 4x}{4}\right)\Bigg|_0^{\pi/2} = \frac{3\pi}{16}.\end{aligned}$$

2. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int (\ln x)^n \, dx \qquad \text{in terms of} \qquad \int (\ln x)^{n-1} \, dx$$

(b) (10 points) Use this formula to find

$$\int (\ln x)^2 \, dx.$$

Solution:

(a) The integration by parts formula is

$$\int u \, dv = uv - \int v \, du.$$

In this case we set

$$\begin{aligned} u &= (\ln x)^n & dv &= dx \\ du &= \frac{n(\ln x)^{n-1} \, dx}{x} & v &= x \end{aligned}$$

so

$$\begin{aligned} \int (\ln x)^n \, dx &= \int u \, dv \; = \; uv - \int v \, du \\ &= x(\ln x)^n - \int x \frac{n(\ln x)^{n-1} \, dx}{x} \\ &= x(\ln x)^n - n \int (\ln x)^{n-1} \, dx. \end{aligned}$$

(b) Put $n = 2$ and $n = 1$ in the result of (a). Then we have

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x dx,$$

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C.$$

Combining these two equations, we have

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2x \ln x + 2x + C.$$

3. (20 points) (a) (10 points) Find the integral

$$\int_{-1}^{0} \frac{dx}{\sqrt{x^2+4x+3}}$$

(b) (10 points) Find the integral

$$\int_{4}^{6} \sqrt{8x-x^2}dx.$$

Solution: (a) We have

$$x^2+4x+3 = (x+2)^2 - 1$$

We use the substitution $x+2 = \sec\theta$. Then we have

$$\begin{aligned} dx &= \sec\theta\tan\theta d\theta \\ \sqrt{x^2+4x+3} &= \tan\theta \end{aligned}$$

so our integral is

$$\begin{aligned} \int_{-1}^{0} \frac{dx}{\sqrt{x^2+4x+3}} &= \int_{0}^{\pi/3} \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta|_{0}^{\pi/3} \\ &= \ln(\sqrt{3}+2) \end{aligned}$$

(b) We have

$$8x - x^2 = 16 - (x-4)^2$$

We use the substitution $x-4 = 4\sin\theta$. From this we get

$$\begin{aligned} dx &= 4\cos\theta d\theta \\ \sqrt{8x-x^2} &= 4\cos\theta \end{aligned}$$

so

$$\begin{aligned}\int_4^6 \sqrt{8x - x^2}dx &= \int_0^{\pi/6} (4\cos\theta)4\cos\theta d\theta \\ &= 16\int_0^{\pi/6} \cos^2\theta d\theta \\ &= 8\int_0^{\pi/6} (1+\cos 2\theta)d\theta \\ &= 8\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/6} \\ &= \frac{4}{3}\pi + 2\sqrt{3}.\end{aligned}$$

4. (20 points) Consider the curve

$$f(x) = 2x^{3/2} + 7$$

(a) (10 points) Calculate the arc length function $s(t)$ starting at $x = 0$, that computes the length of the curve from $(0, f(0))$ to $(t, f(t))$.

(b) (10 points) Calculate the arc length from $x = 2$ to $x = 4$.

Solution: (a) $f'(x) = 3\sqrt{x}$, so the substitution $u = 1 + 9x$ yields

$$\begin{aligned} s(t) &= \int_0^t \sqrt{1 + (3\sqrt{x})^2}\, dx \\ &= \int_0^t \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9}\int_1^{1+9t} \sqrt{u}\, du \\ &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{1+9t} \\ &= \frac{2}{27}(1 + 9t)^{3/2} - \frac{2}{27} \end{aligned}$$

for $t \geq 0$.

(b) By the definition of the arc length function, $s(4)$ is the arc length from $t = 0$ to $t = 4$ and $s(2)$ is the arc length from $t = 0$ to $t = 2$, so the arc length from $t = 2$ to $t = 4$ is

$$s(4) - s(2) = \frac{2}{27}(37)^{3/2} - \frac{2}{27} - \left(\frac{2}{27}(19)^{3/2} - \frac{2}{27}\right) = \frac{74}{27}\sqrt{37} - \frac{38}{27}\sqrt{19}$$

5. (20 points) Consider region between the curves $y = 2x$ and $y = x^2$.

(a) Find the volume of the solid of revolution about the x-axis.

(b) Find the volume of the solid of revolution about the y-axis.

Solution: (a) This is a washer method problem. The region bounded by the two graphs sits between $x = 0$ and $x = 2$ and in that interval $2x$ is the bigger function. We have

$$\begin{aligned} V &= \int_0^2 \pi[(2x)^2 - (x^2)^2]\,dx \\ &= \pi \int_0^2 (4x^2 - x^4)\,dx \\ &= \pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\ &= \pi\left(\frac{32}{3} - \frac{32}{5}\right) \\ &= \frac{64\pi}{15} \end{aligned}$$

(b) This is a shell method problem. We have

$$\begin{aligned} V &= \int_0^2 2\pi x(2x - x^2)\,dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)\,dx \\ &= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\ &= 2\pi\left(\frac{16}{3} - 4\right) \\ &= \frac{8\pi}{3} \end{aligned}$$

OR:

We can integrate with respect to y and use the washer method again:

$$\begin{aligned} V &= \int_0^4 \pi[(\sqrt{y})^2 - (y/2)^2]\,dy \\ &= \pi\int_0^4 [y - (y^2/4)]\,dy \\ &= \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi\left(8 - \frac{16}{3}\right) \\ &= \frac{8\pi}{3} \end{aligned}$$