Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 23, 2007

1. (16 points) Consider the functions $y = x^2$ and $y = 3x$.

(a) Sketch the region enclosed by the graphs of the given functions, and find the area of this region.

(b) Let S be the solid obtained by rotating the above region about the x-axis. Sketch S, along with a typical cross-section of S, and find the volume of S using the washer method (also called the cross-sectional method.)

Solution: (a) A sketch of a similar region is on page 426 in the textbook (page 448 in the 5th edition). The area of the region is

$$A = \int_0^3 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{3 \cdot 3^2}{2} - \frac{3^3}{3} = \frac{27}{6} = \frac{9}{2}.$$

(b) A sketch of a similar solid of revolution is on page 426 in the textbook (448 in the 5th edition). Using washers, the volume is

$$V = \int_0^3 (\pi(3x)^2 - \pi(x^2)^2)\, dx = \pi\left[\frac{9x^3}{3} - \frac{x^5}{5}\right]_0^3 = \pi\left(\frac{9 \cdot 3^3}{3} - \frac{3^5}{5}\right) = \frac{162\pi}{5}.$$

2. (16 points) Again consider the functions $y = x^2$ and $y = 3x$.

(a) Let S be the solid obtained by rotating the region bounded by the graphs of these functions about the y-axis. Sketch S, along with a typical cylindrical shell inside S, and find the volume of S using the cylindrical shells method.

(b) Let S be the solid obtained by rotating the region bounded by the graphs of these functions about the line $x = -3$. Sketch S and find the volume of S using whichever method you want (washer method or cylindrical shells.)

Solution: (a) A similar problem is done in complete detail in your textbook on page 435 (page 457 of the 5th edition), so we only give the answer here.

$$\begin{aligned} V &= \int_0^3 (2\pi x)(3x - x^2)\,dx = 2\pi \int_0^3 (3x^2 - x^3)\,dx \\ &= 2\pi \left[x^3 - \frac{x^4}{4}\right]_0^3 = 2\pi\left(27 - \frac{81}{4}\right) = \frac{27\pi}{2} \end{aligned}$$

(b) A similar problem is done using washers in your textbook on page 429 (pages 449-450 in the 5th edition), so we only give the answer here.

$$\begin{aligned} V &= \int_0^9 \left[\pi(3 + \sqrt{y})^2 - \pi\left(3 + \frac{y}{3}\right)^2\right] dy \\ &= \pi \int_0^9 \left(9 + 6\sqrt{y} + y - 9 - 2y - \frac{y^2}{9}\right) dy \\ &= \pi \int_0^9 \left(6\sqrt{y} - y - \frac{y^2}{9}\right) dy \\ &= \pi \left[4y^{3/2} - \frac{y^2}{2} - \frac{y^3}{27}\right]_0^9 = \pi\left(108 - \frac{81}{2} - 27\right) = \frac{81\pi}{2}. \end{aligned}$$

Using cylindrical shells, the radius is $(3 + x)$, the height is $(3x - x^2)$, and the volume is

$$\begin{aligned} V &= \int_0^3 2\pi(3 + x)\left(3x - x^2\right)\,dx = 2\pi\int_0^3 \left(9x - x^3\right)\,dx \\ &= 2\pi\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2}. \end{aligned}$$

3. (10 points) A rectangular swimming pool is 10 meters long and 4 meters wide, the sides are 2 meters high and the depth of the water is 1.5 meters. How much work is required to pump out all the water over the side? (Note: Use $g = 9.8m/s^2$ as the acceleration due to gravity and 1000 kg/m^3 as the desnity of water. Remember that 1 Joule = 1 $kg\frac{m^2}{s^2}$.)

Solution: Let y_i^* be the height of the ith layer of water, $0 \leq y_i^* \leq 1.5$. An approximation to volume of the ith layer of water is

$$V_i \approx (\text{area})(\text{thickness}) = 40\Delta y.$$

The ith layer of water must travel a vertical distance of $2 - y_i^*$ to the top of the tank. Thus, the amount of work done pumping the water out of the tank is

$$\begin{aligned}
W &= \lim_{n\to\infty} \sum_{i=1}^{n} (\text{density})(\text{volume})(\text{acceleration})(\text{distance}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (1000)(40\Delta y)(9.8)(2 - y_i^*) \\
&= 392000 \int_0^{1.5} (2 - y)\, dy \\
&= 392000 \left[2y - \frac{y^2}{2}\right]_0^{1.5} \\
&= 392000 \left(3 - \frac{9}{8}\right) \\
&= 392000 \frac{15}{8} \\
&= 735000 \quad \text{Joules.}
\end{aligned}$$

One could also define y to be the diatance from the top of the pool rrtaher than the bottom. Then theintegarl above would be replaced by

$$\int_{.5}^{2} y\, dy = \left[\frac{y^2}{2}\right]_{.5}^{2} = 2 - \frac{1}{8} = \frac{15}{8},$$

giving the answer as before.

4. (15 points) Evaluate the following integrals:

(a) $\int x^2 \cos(x^3 + 26)\, dx$

(b) $\int_e^{2e} \frac{1}{x(\ln x)^3}\, dx$

(c) $\int x^5 \sqrt{1 + x^2}\, dx$

Solution: (a) Let $u = x^3 + 26$, then $du = 3x^2\, dx$ and

$$\begin{aligned} \int x^2 \cos(x^3 + 26)\, dx &= \frac{1}{3} \int \cos(u)\, du \\ &= \frac{1}{3}[\sin(u)] + C \\ &= \frac{\sin(x^3 + 26)}{3} + C \end{aligned}$$

Solution: (b) Let $u = \ln x$, then $du = \frac{1}{x}\, dx$ and when $x = 2e$, $u = \ln(2e) = 1 + \ln(2)$, when $x = e$, $u = 1$, and

$$\begin{aligned} \int_e^{2e} \frac{1}{x(\ln x)^3}\, dx &= \int_1^{1+\ln(2)} \frac{1}{u^3}\, du \\ &= -\frac{1}{2u^2}\Big|_1^{1+\ln(2)} \\ &= -\frac{1}{2(1 + \ln(2))^2} - \left(-\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2(1 + \ln(2))^2} \end{aligned}$$

Solution: (c) Let $u = 1 + x^2$, then $du = 2x\,dx$, $x^2 = u - 1$, and

$$\begin{aligned}
\int x^5 \sqrt{1+x^2}\, dx &= \int x^4 \sqrt{1+x^2}\, x\, dx \\
&= \frac{1}{2} \int (u-1)^2 \sqrt{u}\, du \\
&= \frac{1}{2} \int (u^2 - 2u + 1) u^{1/2}\, du \\
&= \frac{1}{2} \int u^{5/2} - 2u^{3/2} + u^{1/2}\, du \\
&= \frac{1}{2} \left[\frac{2}{7} u^{7/2} - 2 \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + C \\
&= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + C \\
&= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + C \\
&= \frac{(1+x^2)^{\frac{3}{2}} \, (8 - 12\, x^2 + 15\, x^4)}{105} + C
\end{aligned}$$

5. (15 points) Evaluate the following integrals:

(a) $\int x^2 \, e^x \, dx$

(b) $\int x \, \sin x \, dx$

(c) $\int \arctan(2x) \, dx$

Solution: (a) Use integration by parts with $u = x^2$ and $dv = e^x \, dx$, then $du = 2x \, dx$ and $v = e^x$ and

$$\int x^2 \, e^x \, dx = x^2 e^x - 2 \int x \, e^x \, dx$$

Using integration by parts a second time with $u = x$ and $dv = e^x \, dx$, then $du = dx$ and $v = e^x$, the integral becomes:

$$\begin{aligned} x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) &= x^2 e^x - 2[x e^x - e^x] + C \\ &= (x^2 - 2x + 2) e^x + C \end{aligned}$$

Solution: (b) Use integration by parts with $u = x$ and $dv = \sin x \, dx$, then $du = dx$ and $v = -\cos x$ and

$$\begin{aligned} \int x \, \sin x \, dx &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Solution: (c) Use integration by parts with $u = \arctan(2x)$ and $dv = dx$, then $du = \frac{2}{1+(2x)^2}$ and $v = x$ and

$$\begin{aligned} \int \arctan(2x) \, dx &= x \arctan(2x) - 2 \int \frac{x}{1 + (2x)^2} \, dx \\ &= x \arctan(2x) - 2 \int \frac{x}{1 + 4x^2} \, dx \end{aligned}$$

Now using the subsitution $u = 4x^2$, then $du = 8x \, dx$ and the integral becomes:

$$\begin{aligned} x \arctan(2x) - \frac{1}{4} \int \frac{1}{1+u} \, du &= x \arctan(2x) - \frac{1}{4} \ln|1 + u| + C \\ &= x \arctan(2x) - \frac{1}{4} \ln|1 + 4x^2| + C \end{aligned}$$

6. (14 points) Evaluate the following integrals:

(a) $\displaystyle\int_0^{10} \frac{1}{\sqrt[3]{x-10}}\,dx$

(b) $\displaystyle\int \sin^5\theta\cos^{10}\theta\,d\theta$

Solution: (a) This is an improper integral:

$$\int_0^{10} \frac{1}{\sqrt[3]{x-10}}\,dx = \lim_{b\to 10^-}\int_0^{b} \frac{1}{\sqrt[3]{x-10}}\,dx$$

Now, let $u = x-10$, then $du = dx$ and when $x = 0$, $u = -10$, when $x = b$, $u = b-10$. Now the integral becomes:

$$\lim_{b\to 10^-}\int_{-10}^{b-10} u^{-1/3}\,du = \lim_{b\to 10^-}\frac{3}{2}u^{2/3}\Big|_{-10}^{b-10} = \lim_{b\to 10^-}\frac{3}{2}\left[(b-10)^{2/3} - (-10)^{2/3}\right] = -\frac{3}{2}10^{2/3}$$

Solution: (b) Notice that the power of sin is odd, so we factor out $\sin\theta$ and write everything in terms of $\cos\theta$:

$$\begin{aligned}\int \sin^5\theta\cos^{10}\theta\,d\theta &= \int \sin^4\theta\cos^{10}\theta\sin\theta\,d\theta\\ &= \int (\sin^2\theta)^2\cos^{10}\theta\sin\theta\,d\theta\\ &= \int (1-\cos^2\theta)^2\cos^{10}\theta\sin\theta\,d\theta\end{aligned}$$

Now let $u = \cos\theta$, then $du = -\sin\theta\,d\theta$ and

$$\begin{aligned}\int \sin^5\theta\cos^{10}\theta\,d\theta &= -\int (1-u^2)^2u^{10}du\\ &= -\int (1-2u^2+u^4)u^{10}du\\ &= -\int (u^{10}-2u^{12}+u^{14})du\\ &= -\left[\frac{u^{11}}{11} - \frac{2u^{13}}{13} + \frac{u^{15}}{15}\right] + C\\ &= -\left[\frac{\cos^{11}\theta}{11} - \frac{2\cos^{13}\theta}{13} + \frac{\cos^{15}\theta}{15}\right] + C\\ &= -\frac{\cos^{11}\theta}{11} + \frac{2\cos^{13}\theta}{13} - \frac{\cos^{15}\theta}{15} + C\end{aligned}$$

7. (14 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{49+x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{x^2+8x+15}\,dx$

Solution: (a) Use the trigonometric subsitution $x = 7\tan\theta$, then $d\theta = 7\sec^2\theta$, and

$$\begin{aligned}
\int \frac{1}{\sqrt{49+x^2}}\,dx &= \int \frac{1}{\sqrt{49+49\tan^2\theta}}\,7\sec^2\theta d\theta \\
&= \int \frac{1}{\sqrt{49(1+\tan^2\theta)}}\,7\sec^2\theta d\theta \\
&= \int \frac{1}{\sec\theta}\,\sec^2\theta d\theta \\
&= \int \sec\theta d\theta \\
&= \int \sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta
\end{aligned}$$

Now let $u = \sec\theta + \tan\theta$, then $du = \sec^2\theta + \sec\theta\tan\theta$ and

$$\begin{aligned}
\int \frac{1}{\sqrt{49+x^2}}\,dx &= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\sec\theta+\tan\theta| + C
\end{aligned}$$

And since $x = 7\tan\theta$, then $\tan\theta = \frac{x}{7}$ and using a right triangle (or $\tan^2\theta + 1 = \sec^2\theta$) we get $\sec\theta = \frac{\sqrt{x^2+49}}{7}$. Now:

$$\begin{aligned}
\int \frac{1}{\sqrt{49+x^2}}\,dx &= \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\frac{\sqrt{x^2+49}}{7} + \frac{x}{7}\right| + C
\end{aligned}$$

Solution: (b) Factorizing the denominator we can re-write the integral

$$\int \frac{1}{x^2+8x+15}\,dx = \int \frac{1}{(x+3)(x+5)}\,dx$$

Now using partial fractions:

$$\frac{1}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5} = \frac{A(x+5)+B(x+3)}{(x+3)(x+5)}$$

Since the denominators are the same we can find A and B by setting the numerators equal to each other:

$$1 = A(x+5) + B(x+3)$$

This equation holds for all real numbers in particular for $x = -5$ we get $B = -\frac{1}{2}$ and for $x = -3$ we get $A = \frac{1}{2}$. Now the integral becomes:

$$\begin{aligned}
\int \frac{1}{x^2+8x+15}\,dx &= \int \frac{1}{2(x+3)} - \frac{1}{2(x+5)}\,dx \\
&= \frac{1}{2}\int \frac{1}{x+3}\,dx - \frac{1}{2}\int \frac{1}{x+5}\,dx \\
&= \frac{1}{2}\ln|x+3| - \frac{1}{2}\ln|x+5| + C
\end{aligned}$$