Math 162: Calculus IIA

First Midterm Exam Solutions

October 21, 2008

Part A

1. (13 points) Consider the curves described by $y = x^2$ and $y = \sqrt{x}$.

(a) Sketch the region enclosed by these curves, and find the area of this region.

(b) Let S be the solid obtained by rotating the above region about the x-axis. Sketch S, along with a typical cross-section of S, and find the volume of S using the washer method (also called the cross-sectional method.)

Solution: (a) The area of the region is

$$A = \int_0^1 (x^{\frac{1}{2}} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

(b) Using washers, the volume is

$$V = \int_0^1 \pi(\sqrt{x})^2 - \pi(x^2)^2\,dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{\frac{3}{10}\pi}$$

2. (13 points) Again consider the curves described by $y = x^2$ and $y = \sqrt{x}$.

(a) Let S be the solid obtained by rotating the region bounded by these curves about the y-axis. Sketch S, along with a typical cylindrical shell inside S, and find the volume of S using the cylindrical shells method.

(b) Let S be the solid obtained by rotating the region bounded by these curves about the line $x = 4$. Sketch S and find the volume of S using whichever method you want (washer method or cylindrical shells.)

Solution: (a) Using cylindrical shells, the volume of the solid is:

$$\begin{aligned} V &= \int_0^1 (2\pi x)(\sqrt{x} - x^2)\,dx = 2\pi \int_0^1 (x^{\frac{3}{2}} - x^3)\,dx \\ &= 2\pi\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) = \boxed{\frac{3}{10}\cdot\pi} \end{aligned}$$

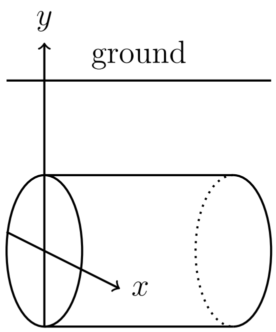
(b) Using cylindrical shells, the radius is $4-x$ and the height is $\sqrt{x}-x^2$, and the volume of the solid is:

$$\begin{aligned} V &= \int_0^1 2\pi(4-x)(\sqrt{x}-x^2)\,dx = 8\pi\int_0^1 \sqrt{x}-x^2\,dx - \int_0^1 2\pi x(\sqrt{x}-x^2)\,dx \\ &= 8\pi\left(\frac{1}{3}\right) - \frac{3}{10}\pi = \boxed{\frac{71}{30}\pi} \end{aligned}$$

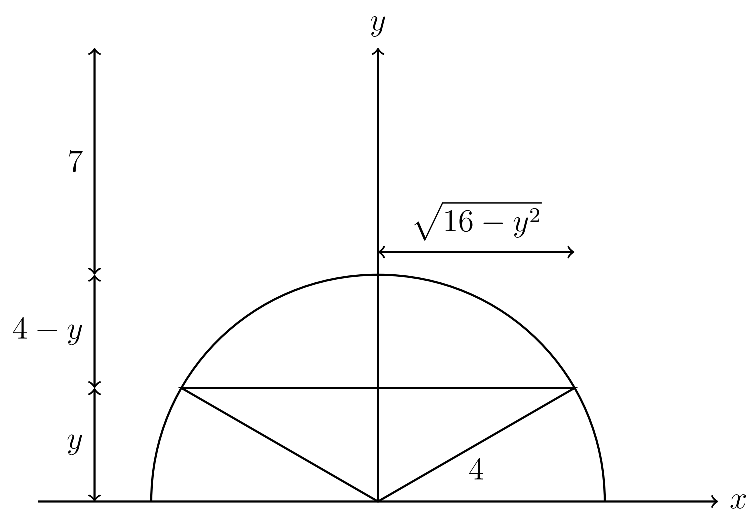
Note that the two integrals in the previous sum were calculated previously.

3. (13 points)

Gasoline at a service station is stored in a cylindrical tank buried on its side, with the highest part of the tank 5 ft below the surface. The tank is 8 feet in diameter and 10 ft long. The density of gasoline is 45 lb/ft^3. Assume that the filler cap of each automobile is 2 feet above the ground. If the tank is initially full, how much work is done pumping half of the gasoline in the tank into automobiles?



Solution: Consider the following picture of the cross-section of the top half of the tank.



Let W_i be the work required to lift the i-th layer, which is at height y. With the choices made in the picture, $0 \le y \le 4$. Then $W_i \approx F_i \cdot d_i$. In this case the displacement of the i-th layer: $d_i = (4-y)+7 = 11-y \quad$ ft, and the weight of the i-th layer is given by $F_i = V_i \cdot 45 \quad$ lbs, where V_i is the volume of the slice of gasoline at height y:

$$V_i = (length)(width)(thickness) = (10)(2\sqrt{16-y^2})(\Delta y)$$

Thus $W_i = (20\sqrt{16-y^2})(45)(7+(4-y))\,\Delta y$.

and thus the amount of work done pumping half of the gasoline out of the tank is

$$\begin{aligned}
W &= \textstyle\int_0^4 (20\sqrt{16-y^2})(45)(7+(4-y))\,dy \\
&= 20\cdot 45 \int_0^4 ((11-y)\sqrt{16-y^2})\,dy \\
&= 20\cdot 45\cdot 11 \int_0^4 \sqrt{16-y^2}\,dy + 20\cdot 45\cdot \frac{1}{2}\int_0^4 -2y\sqrt{16-y^2}\,dy \\
&\quad (\text{Let } u = 16-y^2,\ du = -2y) \\
&= 20\cdot 45\cdot 11\frac{\pi\cdot 4^2}{4} + 10\cdot 45\int_{16}^0 \sqrt{u}\,du \\
&= 20\cdot 45\cdot 11\pi\cdot 4 + 10\cdot 45\left[\frac{2}{3}u^{3/2}\right]_{16}^0 \\
&= 20\cdot 45\cdot 11\pi\cdot 4 - 10\cdot 45\cdot\frac{2}{3}(16)^{3/2} \\
&= \boxed{39,600\pi - 19,200 \text{ ft-lb}}
\end{aligned}$$

4. (12 points)

Find the definite integrals, if they exist:

(a)

$$\int_{-\infty}^{-1} e^{-2t}\,dt$$

(b)

$$\int_{-1}^{1} \frac{1}{x^2-2x}\,dx$$

Solution: (a)

$$\begin{aligned}\int_{-\infty}^{-1} e^{-2t}\,dt &= \lim_{x\to-\infty}\int_{x}^{-1} e^{-2t}\,dt \\ &= \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} \\ &= \lim_{x\to-\infty}\left[-\frac{1}{2}e^{2}+\frac{1}{2}e^{-2x}\right] \\ &= \infty\end{aligned}$$

Thus the integral is divergent.

(b)

$$I=\int_{-1}^{1}\frac{dx}{x^2-2x}=\int_{-1}^{1}\frac{dx}{x(x-2)}=\int_{-1}^{0}\frac{dx}{x(x-2)}+\int_{0}^{1}\frac{dx}{x(x-2)}=I_1+I_2$$

Now:

$$\frac{1}{x(x-2)}=\frac{A}{x}+\frac{B}{x-2}$$

Then $1=A(x-2)+Bx$, and set $x=2$ to get that $B=\frac{1}{2}$ and set $x=0$ to get $A=-\frac{1}{2}$. Therefore,

$$\begin{aligned}I_2 &= \lim_{t\to0^+}\int_t^1\left(\frac{-\frac{1}{2}}{x}+\frac{\frac{1}{2}}{x-2}\right)dx \\ &= \lim_{t\to0^+}\left[-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|x-2|\right]_t^1 \\ &= \lim_{t\to0^+}\left[(0+0)-\left(-\frac{1}{2}\ln t+\frac{1}{2}\ln|t-2|\right)\right] \\ &= -\infty\end{aligned}$$

Since I_2 diverges, I diverges.

5. (20 points) Suppose $f(x)$ is a function whose derivative is given by

$$f'(x)=\sqrt{2x-x^2}$$

(a) Set up an integral for the length of the curve traced out by the graph of $f(x)$ from $x=1$ to $x=2$.

(b) Evaluate the integral found in part (a).

Solution: (a) The integral computing the length of a curve traced out by the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Here $f'(x)$ is given, as are the bounds:

$$\begin{aligned} s &= \int_1^2 \sqrt{1 + (\sqrt{2x - x^2})^2}\, dx \\ &= \boxed{\int_1^2 \sqrt{1 + 2x - x^2}\, dx} \end{aligned}$$

(b) One needs to apply trigonometric substitution, but one needs to complete the square first:

$$-x^2 + 2x + 1 = -(x-1)^2 + 2$$

The integral is thus:

$$\int_1^2 \sqrt{2 - (x-1)^2}\, dx$$

Let $u = x - 1$ so that $du = dx$ to get the integral:

$$\int_0^1 \sqrt{2 - u^2}\, du$$

Now, we need to apply trigonometric substitution. Let $u = \sqrt{2}\sin\theta$ so that $du = \sqrt{2}\cos\theta d\theta$. Substituting these values (and simplifying via Pythagorean trigonometric identities) we get:

$$\begin{aligned} \int_0^1 \sqrt{2 - u^2}\, du &= \int_0^{\pi/4} \sqrt{2 - 2\sin^2\theta}(\sqrt{2}\cos\theta d\theta) \\ &= \int_0^{\pi/4} 2\cos^2\theta\, d\theta \end{aligned}$$

Now, we use the half-angle formula:

$$\begin{aligned} &= \int_0^{\pi/4} 1 + \cos 2\theta\, d\theta \\ &= \left(\theta + \frac{1}{2}\sin 2\theta\right) \Big|_0^{\pi/4} \\ &= \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - 0 - \frac{1}{2}\sin 0 \\ &= \boxed{\frac{\pi}{4} + \frac{1}{2}} \end{aligned}$$

6. (16 points) Evaluate the integrals:

(a) $\displaystyle\int \tan x \sec^4 x \, dx =$

(b) $\displaystyle\int x \tan x \sec^4 x \, dx =$

Solution:

(a) Here, we would like to save $\sec x \tan x$ for later, making our choice of $u = \sec x$ and $du = \sec x \tan x dx$:

$$\begin{aligned}\int \tan x \sec^4 x \, dx &= \int \sec^3 x \cdot \sec x \tan x \, dx \\ &= \int u^3 \, du \\ &= \frac{1}{4}u^4 + C \\ &= \boxed{\frac{1}{4}\sec^4 x + C}\end{aligned}$$

(b) This is a product of functions, and we are unable to implement u-substitution. Here, <u>Integration by Parts</u> is needed. Let $u = x$ and $dv = \tan x \sec^4 x \, dx$ so that $du = dx$ and $v = \frac{1}{4}\sec^4 x$.

$$\begin{aligned}\int x \tan x \sec^4 x \, dx &= uv - \int v \, du \\ &= \frac{1}{4}x \sec^4 x - \frac{1}{4}\int \sec^4 x \, dx\end{aligned}$$

Now to evaluate the final integral, we save $\sec^2 x$ for later and let $u = \tan x$:

$$\begin{aligned}\int \sec^4 x \, dx &= \int \sec^2 x \cdot \sec^2 x \, dx \\ &= \int (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int u^2 + 1 \, du \\ &= \frac{1}{3}u^3 + u + C \\ &= \frac{1}{3}\tan^3 x + \tan x + C\end{aligned}$$

Combining this information

$$\begin{aligned}\int x\tan x\sec^4 x\,dx &= \frac{1}{4}x\sec^4 x - \frac{1}{4}\int \sec^4 x\,dx\\ &= \frac{1}{4}x\sec^4 x - \frac{1}{4}\left(\frac{1}{3}\tan^3 x + \tan x\right) + C\\ &= \boxed{\frac{1}{4}x\sec^4 x - \frac{1}{12}\tan^3 x - \frac{1}{4}\tan x + C}\end{aligned}$$

7. (13 points) Find the area of the surface obtained by rotating the curve $y = \sqrt{2x+1}$, $1 \le x \le 7$, about the x-axis. The area of a surface obtained by rotating a curve $y = f(x)$ around the x-axis is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Here $f(x) = \sqrt{2x+1}$ so that the derivative is:

$$f'(x) = \frac{1}{\sqrt{2x+1}}$$

This makes the surface area:

$$\begin{aligned}A &= \int_1^7 2\pi\sqrt{2x+1}\sqrt{1 + \frac{1}{2x+1}}\,dx\\ &= \int_1^7 2\pi\sqrt{2x+1}\sqrt{\frac{2x+1+1}{2x+1}}\,dx\\ &= \int_1^7 2\pi\sqrt{2x+2}\,dx\end{aligned}$$

Let $u = 2x+2$ so that $du = 2dx$:

$$\begin{aligned}\int_1^7 2\pi\sqrt{2x+2}\,dx &= 2\pi\int_4^{16}\sqrt{u}\left(\frac{du}{2}\right)\\ &= \pi\int_4^{16} u^{1/2}\,du\\ &= \pi\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{16}\\ &= \frac{2\pi}{3}\left(16^{3/2} - 4^{3/2}\right)\\ &= \frac{2\pi}{3}(64-8)\\ &= \frac{2\pi}{3}(56)\\ &= \boxed{\frac{112\pi}{3}}\end{aligned}$$