Math 162

First Midterm ANSWERS

October 24, 2006

Answer:

1. (8 points)

Solve the integral

$$\int x^3\sqrt{x^2+3}\,dx$$

Answer:

We use u-substitution, with $u = x^2+3$, so $du = 2xdx$. Then

$$x^3dx = \frac{1}{2}x^2du = \frac{1}{2}(u-3)du$$

so

$$\begin{aligned}
\int x^3\sqrt{x^2+3}\,dx &= \frac{1}{2}\int u^{1/2}(u-3)du \\
&= \frac{1}{2}\int \left(u^{3/2}-3u^{1/2}\right)du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-3\frac{2}{3}u^{3/2}\right)+C \\
&= \frac{1}{5}(x^2+3)^{5/2}-(x^2+3)^{3/2}+C.
\end{aligned}$$

2. (8 points)

Solve the integral

$$\int \frac{1}{(9-4x^2)^{3/2}}\,dx$$

Answer:

We use the trigonometric substitution $x = (3/2)\sin u$, so $dx = (3/2)\cos u\, du$. Then

$$\begin{aligned}
\int \frac{1}{(9-4x^2)^{3/2}}\, dx &= \frac{1}{27}\int \frac{(3/2)\cos u\, du}{(9-9\sin^2 u)^{3/2}} \\
&= \frac{1}{27}\cdot\frac{3}{2}\int \frac{\cos u}{\cos^3 u}\, du \\
&= \frac{1}{18}\int \sec^2 u\, du \\
&= \frac{1}{18}\tan u + C
\end{aligned}$$

Drawing a triangle, we find that

$$\tan u = \frac{2x}{\sqrt{9-4x^2}}$$

and so the answer is

$$\frac{1}{18}\cdot\frac{2x}{\sqrt{9-4x^2}} + C = \frac{x}{9\sqrt{9-4x^2}} + C$$

3. (8 points)

Solve the integral

$$\int \frac{x^2+3x}{x^2+3x+2}\, dx$$

Answer:

Using long division of fractions, we find that

$$\frac{x^2+3x}{x^2+3x+2} = 1 - \frac{2}{x^2+3x+2}$$

Using partial fractions, we find that

$$\frac{2}{x^2+3x+2} = \frac{2}{x+1} - \frac{2}{x+2}$$

and so

$$\begin{aligned}
\int \frac{x^2+3x}{x^2+3x+2}\, dx &= \int dx - \int \frac{2}{x+1}\, dx + \int \frac{2}{x+2}\, dx \\
&= x - 2\ln(x+1) + 2\ln(x+2) + C.
\end{aligned}$$

4. (10 points)

The density of water is 1000 kg/m^3 and the gravitational constant is 9.8 m/s^2.

A tank has a parabolic shape, obtained by rotating the curve $y = x^2$ about the y-axis, between $y = 0$ and $y = 16$. Distance along both axes is measured in meters. The tank is filled with water to the top. How much work does it take to bring all the water to the top of the tank?

Answer:

We use horizontal slices. Each slice is circular, with radius x and thickness Δy. Thus, the volume of a slice is $\pi x^2 \Delta y = \pi y \Delta y$. The mass of a slice is $1000\pi y \Delta y$. The force on a slice is $9800\pi y \Delta y$. This is obtained by multiplying by the gravitational constant 9.8. Finally, the work done to raise a slice to the top is $9800\pi y(16 - y)\Delta y$ since it must be raised a distance $16 - y$. Therefore, the total work is

$$\begin{aligned}
\int_0^{16} 9800\pi y(16 - y)\,dy &= 9800\pi \int_0^{16} (16y - y^2)\,dy \\
&= 9800\pi \left(8y^2 - \frac{1}{3}y^3 \right) \Big|_0^{16} \\
&= 9800\pi \left(8 \cdot 16^2 - \frac{16^3}{3} \right)
\end{aligned}$$

5. (10 points)

Suppose that the region bounded by

$$\begin{aligned}
y &= 0 \\
y &= \sin x \\
x &= 0 \\
x &= \pi
\end{aligned}$$

is rotated about the y-axis. Find the volume of the resulting region.

Answer:

We use the shell method. The radius of a shell will be x, the height of the shell will be $y = \sin x$, and the thickness will be Δx. The volume of a shell will be 2π times the radius times the height times the thickness, namely $2\pi x \sin x \Delta x$. So, the volume will be

$$V = 2\pi \int_0^{\pi} x \sin x \, dx$$

Using integration by parts, with

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

we get

$$\begin{aligned} V &= -2\pi x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx \\ &= 2\pi^2 + \sin x\Big|_0^\pi \\ &= 2\pi^2 \end{aligned}$$

6. (10 points)

It takes 10 ft-lb of work to stretch a spring from its rest position of 3 ft to 5 ft. How much work would it take to stretch it from 5 ft to 8 ft?

Answer:

Using the equation $F = kx$, we can determine the spring constant k. Going from 3 ft to 5 ft is 2 ft beyond the rest position, so the work done is

$$10 = \int_0^2 kx\,dx = k\frac{x^2}{2}\Big|_0^2 = 2k.$$

Therefore, $k = 5$. Thus, the work done stretching the spring from 5 ft to 8 ft, which is from 2 ft to 5 ft beyond the rest position, is

$$\begin{aligned} W &= \int_2^5 kx\,dx \\ &= \int_2^5 5x\,dx \\ &= \frac{5}{2}x^2\Big|_2^5 \\ &= \frac{125}{2} - 10 \\ &= \frac{105}{2} \end{aligned}$$

7. (10 points)

Find the area of the finite region bounded by the curves $y = x^2$ and $y = x + 2$.

Answer:

First we find the intersection points for the two curves. Set the equations equal to get $x^2 = x + 2$. Then $(x-2)(x+1) = 0$ which gives intersection points $(2,4)$ and $(-1,1)$. Note that $x + 2 \geq x^2$ on $[-1,2]$. Then the area between the two curves is given by:

$$\begin{aligned}\int_{-1}^{2} (x + 2 - x^2)dx &= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} \\ &= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\ &= \frac{9}{2}\end{aligned}$$

8. (10 points)

Find the volume of the solid obtained by rotating the region bounded by the curves $x = y^2$, $x = 4$ and $y = 0$ about the x-axis.

Answer:

We slice the solid perpendicular to x-axis and we integrate with respect to x from 0 to 4. Slicing at distance x, we get a circular disk of radius $y = \sqrt{x}$, and therefore of area πx. So the resulting solid has volume:

$$\int_{0}^{4} \pi x \, dx = \left(\pi \frac{x^2}{2}\right)\Bigg|_{0}^{4} = 8\pi.$$

9. (8 points)

Evaluate the indefinite integral

$$\int e^{\sqrt{x}} dx$$

Answer:

First we use the substitution $y = \sqrt{x}$. Then $dy = \dfrac{dx}{2\sqrt{x}}$ and $dx = 2y\, dy$. So our integral becomes

$$\int e^{\sqrt{x}} dx = \int 2ye^{y}\, dy$$

Now we use integration by parts with

$$\begin{aligned} u &= 2y \qquad & dv &= e^y\, dy \\ du &= 2\, dy \qquad & v &= e^y \end{aligned}$$

$$\begin{aligned}\int 2ye^y\,dy &= 2ye^y - \int 2e^y\,dy \\ &= 2ye^y - 2e^y + C\end{aligned}$$

And substituting back in for y,

$$\int e^{\sqrt{x}}dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

10. (8 points)

Solve this integral

$$\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

Answer:

Since the power of $\sin\theta$ is odd, we keep one $\sin\theta$ and we use the identity $\sin^2\theta + \cos^2\theta = 1$ to write $\sin^2\theta$ in terms of $\cos\theta$:

$$\begin{aligned}\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta &= \int_0^{\pi/2} \cos^2\theta(1-\cos^2\theta)\sin\theta\,d\theta \\ &= \int_0^{\pi/2} (\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta\end{aligned}$$

Let $u = \cos\theta$ and then $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$ and when $\theta = \pi/2$, $u = 0$. The integral becomes

$$\begin{aligned}\int_1^0 (u^2 - u^4)(-du) &= \int_0^1 (u^2-u^4)du \\ &= \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\bigg|_0^1 \\ &= \frac{1}{3} - \frac{1}{5} \\ &= \frac{2}{15}\end{aligned}$$

11. (10 points)

Determine whether the integral

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

is divergent or convergent. For full credit, be sure to explain your reasoning. If it is convergent, evaluate it. If not, state your answer as "divergent."

Answer:

We use integration by parts with

$$\begin{aligned} u &= \ln x & dv &= \frac{1}{x^2}\,dx \\ du &= \frac{1}{x}\,dx & v &= -\frac{1}{x} \end{aligned}$$

Then

$$\begin{aligned} \int \frac{\ln x}{x^2} dx &= -\frac{\ln x}{x} + \int \frac{dx}{x^2} \\ &= -\frac{\ln x}{x} - \frac{1}{x} \\ &= -\frac{\ln x + 1}{x} \end{aligned}$$

Using L'Hopital's rule, we get

$$\begin{aligned} \int_1^\infty \frac{\ln x}{x^2} &= \frac{\ln 1 + 1}{1} - \lim_{t\to\infty} \frac{\ln t + 1}{t} \\ &\stackrel{LH}{=} 1 - \lim_{t\to\infty} \frac{1/t}{1} \\ &= 1 \end{aligned}$$