MATH 162

Midterm Exam 1 - Solutions

February 22, 2007

1. (18 points) Evaluate the following integrals:

(a) $\displaystyle\int x^3 \sin(x^4+7)\,dx$

Solution: Let $u = x^4 + 7$, then $du = 4x^3\,dx$ and

$$\int x^3 \sin(x^4+7)\,dx = \frac{1}{4}\int \sin(u)\,du = \frac{1}{4}\left[-\cos(u)\right] + C = \frac{-\cos(x^4+7)}{4} + C$$

(b) $\displaystyle\int_e^{e^3} \frac{1}{x(\ln x)^2}\,dx$

Solution: Let $u = \ln x$, then $du = \frac{1}{x}\,dx$. If $x = e$ then $u = \ln(e) = 1$, and if $x = e^3$ then $u = \ln(e^3) = 3$ and

$$\int_e^{e^3} \frac{1}{x(\ln x)^2}\,dx = \int_1^3 \frac{1}{u^2}\,du = \left[\frac{-1}{u}\right]_1^3 = \left[\frac{-1}{3} - (-1)\right] = \frac{2}{3}$$

(c) $\displaystyle\int x^5 \sqrt[4]{x^3+2}\,dx$

Solution: Let $u = x^3 + 2$, then $du = 3x^2\,dx$ and $x^3 = u - 2$. Thus

$$\int x^5 \sqrt[4]{x^3+2}\,dx = \int x^3\,(x^3+2)^{1/4}\,x^2\,dx = \frac{1}{3}\int (u-2)\,(u)^{1/4}\,du$$

$$= \frac{4}{27}\,(x^3+2)^{9/4} - \frac{8}{15}\,(x^3+2)^{5/4} + C$$

2. (18 points) Evaluate the following integrals:

(a) $\displaystyle\int x\, e^x\, dx$

Solution: Use integration by parts and let $u = x$ and $dv = e^x\, dx$, then $du = dx$ and $v = e^x$ and

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$$

(b) $\displaystyle\int x^2\, \sin x\, dx$

Solution: Use integration by parts and let $u = x^2$ and $dv = \sin x\, dx$, then $du = 2x\, dx$ and $v = -\cos x$ and

$$\int x^2\, \sin x\, dx = -x^2 \cos x + 2 \int x \cos x\, dx$$

Using integration by parts a second time with $u = x$ and $dv = \cos x\, dx$, then $du = dx$ and $v = \sin x$, the integral becomes

$$= -x^2 \cos x + 2 \int x \cos x\, dx = -x^2 \cos x + 2(x \sin x - \int \sin x\, dx)$$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

(c) $\displaystyle\int \ln(2x+1)\, dx$

Solution: Start with a substitution and let $w = 2x+1$, then $dw = 2\, dx$ and $\displaystyle\int \ln(2x+1)\, dx = \frac{1}{2}\int \ln(w)\, dw$. Now use integration by parts and let $u = \ln w$ and $dv = dw$, then $du = \dfrac{1}{w}\, dw$ and $v = x$ and the integral becomes

$$\int \ln(2x+1)\, dx = \frac{1}{2} \int \ln(w)\, dw = \frac{1}{2}(w \ln w - \int 1\, dw) = \frac{1}{2}(w \ln w - w) + C$$
$$= \frac{1}{2}((2x+1)\ln(2x+1) - (2x+1)) + C$$

(You could also do integration by parts first with $u = \ln(2x+1)$ and $dv = dx$, and then use substitution on the resulting integral.)

3. (12 points) Evaluate the following integrals:

(a) $\int \tan^2\theta \sec^4\theta \, d\theta$

Solution: Notice that the power of secant is even, so we factor out $\sec^2\theta$ and write everything else in terms of $\tan\theta$.

$$\int \tan^2\theta \sec^4\theta \, d\theta = \int \tan^2\theta \sec^2\theta \sec^2\theta \, d\theta = \int \tan^2\theta(1+\tan^2\theta)\sec^2\theta \, d\theta$$

Now let $u = \tan\theta$ and then $du = \sec^2\theta d\theta$ and the integral becomes

$$= \int u^2(1+u^2)\, du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C$$

(b) $\int \sin^4\theta \cos^5\theta \, d\theta$

Solution: Notice that the power of cosine is odd, so we factor out $\cos\theta$ and write everything else in terms of $\sin\theta$.

$$\int \sin^4\theta \cos^5\theta \, d\theta = \int \sin^4\theta \cos^4\theta \cos\theta \, d\theta = \int \sin^4\theta(1-\sin^2\theta)^2\cos\theta \, d\theta$$

Now let $u = \sin\theta$ and then $du = \cos\theta d\theta$ and the integral becomes

$$= \int u^4(1-u^2)^2 \, du = \int (u^4 - 2u^6 + u^8)\, du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\sin^5\theta}{5} - \frac{2\sin^7\theta}{7} + \frac{\sin^9\theta}{9} + C$$

4. (10 points) Evaluate the following integrals.

(a) $\int \tan x \, dx$

Solution: Start by rewriting $\tan x$ as $\frac{\sin x}{\cos x}$, then use substitution by letting $u = \cos x$ and $du = -\sin x \, dx$ and the integral becomes

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{-1}{u} \, du = - \ln |u| + C$$

$$= - \ln |\cos x| + C \qquad \text{OR} \qquad = \ln |\sec x| + C$$

(Either of these last two answers is correct.)

(b) $\int \arctan x \, dx$

Solution: Use integration by parts and let $u = \arctan x$ and $dv = dx$, then $du = \dfrac{1}{x^2+1} \, dx$ and $v = x$ and

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2+1} \, dx$$

Now use substitution with $w = x^2 + 1$ and $dw = 2x \, dx$ and the integral becomes

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \int \frac{1}{w} \, dw = x \arctan x - \frac{1}{2} \ln |w| + C$$

$$= x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C$$

5. (16 points) Consider the functions $y = x^2$ and $y = x$.

(a) Sketch the region enclosed by the graphs of the given functions, and find the area of this region.

Solution: A sketch of the region is on page 448 in the textbook. The area of the region is

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1^2}{2} - \frac{1^3}{3} = \frac{1}{6}.$$

(b) Let S be the solid obtained by rotating the above region about the x-axis. Sketch S, along with a typical cross-section of S, and find the volume of S using the washer method (also called the cross-sectional method.)

Solution: A sketch of the solid of revolution is on page 448 in the textbook. Using washers, the volume is

$$V = \int_0^1 (\pi(x)^2 - \pi(x^2)^2)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1^3}{3} - \frac{1^5}{5}\right) = \frac{2\pi}{15}.$$

6. (16 points) Again consider the functions $y = x^2$ and $y = x$.

(a) Let S be the solid obtained by rotating the region bounded by the graphs of these functions about the y-axis. Sketch S, along with a typical cylindrical shell inside S, and find the volume of S using the cylindrical shells method.

Solution: This is done in complete detail in your textbook on page 457, so we only give the answer here.

$$V = \int_0^1 (2\pi x)(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}.$$

(b) Let S be the solid obtained by rotating the region bounded by the graphs of these functions about the line $x = -1$. Sketch S and find the volume of S using whichever method you want (washer method or cylindrical shells.)

Solution: This problem is done using washers in your textbook on pages 449-450, so we only give the answer here.

$$V = \int_0^1 \left[\pi(1+\sqrt{y})^2 - \pi(1+y)^2\right]\, dy = \pi \int_0^1 (2\sqrt{y} - y - y^2)\, dy = \pi \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{2}.$$

Using cylindrical shells, the radius is $(1 + x)$, the height is $(x - x^2)$, and the volume is

$$V = \int_0^1 2\pi(1+x)(x - x^2)\, dx = 2\pi \int_0^1 (x - x^3)\, dx = 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}.$$

7. (10 points) A trough (to be attached to a wall) is 4 meters long, and the vertical cross-sections of the trough parallel to an end are shaped like a right triangle (with height 2 meters and top of length 1 meters). The trough is filled to a height of 1 meter with water (density 1000 kg/m^3). Find the amount of work (in joules) required to empty the trough by pumping the water over the top. (Note: Use $g = 9.8m/s^2$ as the acceleration due to gravity. Remember that 1 Joule = 1 $kg\frac{m^2}{s^2}$.)

Solution: Place the side of the triangle of length 2 along the y-axis and the hypoteneuse of the triangle on the line $y = 2x$. Let y_i^* be the height of the ith layer of water, $0 \leq y_i^* \leq 1$. An approximation to volume of the ith layer of water is

$$V_i \approx (\text{length})(\text{width})(\text{thickness}) = 4x_i^*\Delta y = 4(y_i^*/2)\Delta y = 2y_i^*\Delta y.$$

The ith layer of water must travel a vertical distance of $2 - y_i^*$ to the top of the tank. Thus, the amount of work done pumping the water out of the tank is

$$\begin{aligned}
W &= \lim_{n\to\infty} \sum_{i=1}^{n} (\text{density})(\text{volume})(\text{acceleration})(\text{distance}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (1000)(2y_i^*\Delta y)(9.8)(2 - y_i^*) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (2000)(9.8)(2y_i^* - (y_i^*)^2)\Delta y \\
&= 19600 \int_0^1 (2y - y^2)\, dy \\
&= 19600 \left[y^2 - \frac{y^3}{3} \right]_0^1 \\
&= 19600 \left(1^2 - \frac{1^3}{3} \right) \\
&= \frac{39200}{3} \quad J.
\end{aligned}$$