Math 162: Calculus IIA

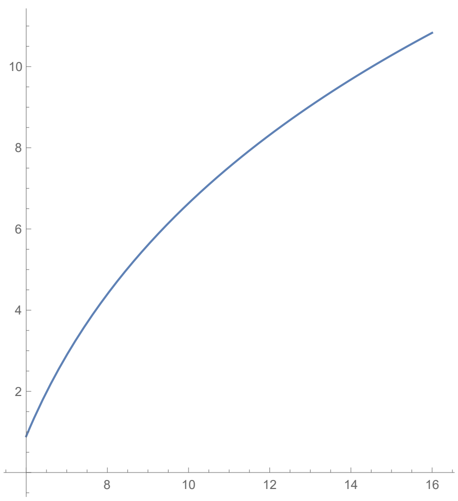
Final Exam ANSWERS

May 8, 2019

Part A

1. (20 points)

Find the arc length L of the parametric curve, $x = 2t$, $y = 4\ln((t/2)^2 - 1)$, from $t = 6$ to $t = 7$.



Answer:

$dx/dt = 2$, $dy/dt = 8t/(t^2 - 4)$.

So

$$(dx/dt)^2 + (dy/dt)^2 = 4 + (64t^2)/(t^2 - 4)^2.$$

Hence

$(dx/dt)^2 + (dy/dt)^2 = 4 + (64t^2)/(t^2 - 4)^2) = (4t^4 + 32t^2 + 64)/(t^2 - 4)^2 = 4(t^2 + 4)^2/(t^2 - 4)^2.$

Therefore

$$\sqrt{(dx/dt)^2 + (dy/dt)^2} = 2(t^2 + 4)/(t^2 - 4) = 2 + 16/(t^2 - 4).$$

Using partial fractions, $16/(t^2 - 4) = -4/(t + 2) + 4/(t - 2)$, so

$$\sqrt{(dx/dt)^2 + (dy/dt)^2} = 2 - 4/(t + 2) + 4/(t - 2),$$

and

$$L = \int_6^7 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_6^7 (2 - 4/(t+2) + 4/(t-2)) dt$$
$$= [2t - 4\ln(t+2) + 4\ln(t-2)]_6^7 = 2 - 4\ln 9 + 4\ln 8 + 4\ln 5 - 4\ln 4$$
$$= 2 + 4\ln(10/9)$$

2. (20 points) Compute

$$\int \frac{1}{\sqrt{1+(6x-4)^2)}} dx$$

Answer:

Make the trig substitution $6x - 4 = \tan(\theta)$. Then $6dx = \sec^2(\theta)d\theta$ and $x = (\tan\theta + 4)/6$. So

$$\int \frac{1}{\sqrt{1+(6x-4)^2)}} dx = 1/6 \int \frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = 1/6 \int \frac{\sec^2\theta d\theta}{\sec\theta}$$
$$= 1/6 \int \sec\theta d\theta = 1/6 \ln|\sec\theta + \tan\theta|.$$

We have that $\tan\theta = 6x - 4$, so

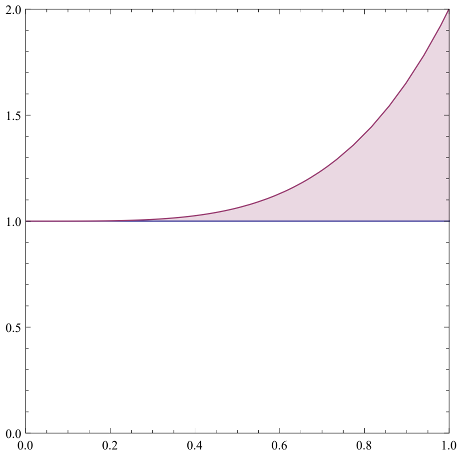
$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(6x-4)^2} = \sqrt{36x^2 - 48x + 17}.$$

Hence

$$\int \frac{1}{\sqrt{1+(6x-4)^2)}} dx = 1/6 \ln|6x - 4 + \sqrt{36x^2-48x+17}| + C.$$

3. (20 points)

(a) Compute the volume of a region bounded by the curves $y = x^4 + 1$, $y = 1$ and $x = 1$ and rotated around the y-axis.

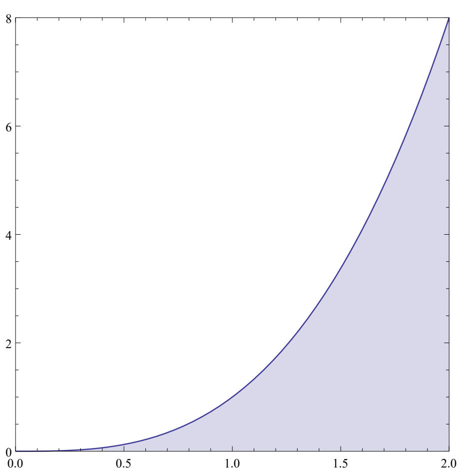


Answer:

Using the shell method we have shells of radius x, thickness dx and height $(x^4+1)-1=x^4$. Therefore

$$V=\int_0^1 2\pi x\cdot x^4 dx = 2\pi \frac{x^6}{6}\Big|_0^1 = \frac{\pi}{3}$$

(b) Set up the integral for the volume of the region bounded by $y=x^3$, $y=0$ and $x=2$ and rotated around line $x=2$. Use the shell method. Do not evaluate the integral.



Answer:

Using the shell method we have shells of radius $(2-x)$, thickness dx and height x^3. Thus the volume is

$$V=\int_0^2 2\pi(2-x)x^3\,dx.$$

4. (10 points)

Evaluate the integral

$$\int \arctan(2x)dx.$$

Answer:

Using integration by parts with $u = \arctan(2x)$ and $dv = dx$ yields $du = \frac{2}{1+4x^2}$ and $v = x$, so we have

$$\int \arctan(2x)dx = x\arctan(2x) - \int \frac{2x}{1+4x^2}dx$$

then a substitution of $w = 1 + 4x^2$, $dw = 8xdx$ yields

$$\int \frac{2x}{1+4x^2}dx = \frac{1}{4}\int \frac{dw}{w} = \frac{1}{4}\ln|w| - C = \frac{1}{4}\ln(1+4x^2) - C$$

thus

$$\int \arctan(2x)dx = x\arctan(2x) - \frac{1}{4}\ln(1+4x^2) + C.$$

5. (20 points)

(a) Find the partial fraction decomposition of

$$\frac{x^2+3x}{x^2-4}.$$

Answer:

The fraction is improper so first use long division to write:

$$\frac{x^2+3x}{x^2-4} = 1 + \frac{3x+4}{x^2-4}.$$

Since the denominator is a difference of squares $x^2 - 4 = (x-2)(x+2)$ we next seek constants A, B such that:

$$\frac{3x+4}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

which is equivalent to solving the linear system:

$$A + B = 3$$
$$2A - 2B = 4$$

Adding the first equation to half of the second gives $2A = 5$ so $A = 5/2$ and therefore $B = 1/2$. Thus:

$$\frac{x^3 + 3x}{x^2 - 4} = 1 + \frac{5/2}{x-2} + \frac{1/2}{x+2} = 1 + \frac{5}{2x-4} + \frac{1}{2x+4}.$$

(b) Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 - 5}{(x+1)^3(x^2+4)^2(x-1)} = \rule{7cm}{0.4pt}$$

Do not determine the numerical values of the coefficients.

Answer:

All the factors are linear except $x^2 + 4$, which has discriminant $b^2 - 4ac = -16 < 0$ (has complex roots $\pm 2i$) so does not factor over the real numbers. Thus there is a linear factor of multiplicity 3, an irreducible quadratic factor of multiplicity 2 and a linear factor of multiplicity 1. So the partial fraction decomposition will look like:

$$\frac{x^3 - 2}{(x+1)^3(x^2+1)^2(x-1)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3} + \frac{B_1 x + C_1}{x^2+4} + \frac{B_2 x + C_2}{(x^2+4)^2} + \frac{D}{x-1}.$$

(c) Let

$$f(x) = \frac{1}{x-1} + \frac{2x+3}{x^2+1}.$$

Evaluate

$$\int f(x)dx.$$

Answer:

Split the integral:

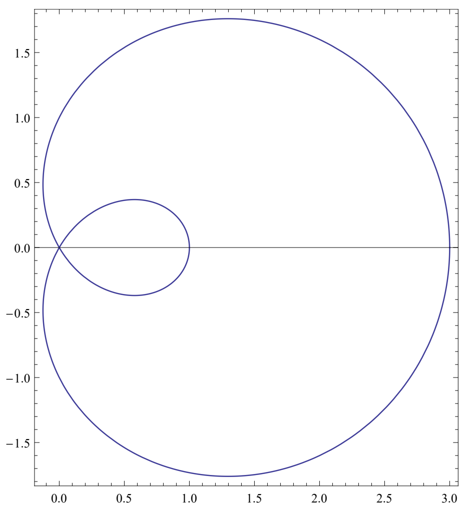
$$\int f(x)dx = \int \frac{1}{x-1}dx + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx$$
$$= \ln|x-1| + \int \frac{2x}{x^2+1}dx + 3\arctan x$$

Substitute $u = x^2+1$ and hence $du = 2xdx$ to get:

$$\int f(x)dx = \ln|x-1| + \ln|x^2+1| + 3\arctan(x) + C.$$

6. (15 points)

Find the area inside the outer (larger) loop but outside the inner (smaller) loop of the limaçon $r = 1 + 2\cos(\theta)$.



Answer:

The curve intersects itself when the radius equals zero, or $2\cos(\theta) = -1$, which means $\cos(\theta) = \frac{-1}{2}$. We know $\cos^{-1}(\frac{-1}{2}) = \frac{2\pi}{3}$ so the points of intersection are $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{4\pi}{3}$. The outer loop is traced out from $\frac{-2\pi}{3}$ to $\frac{2\pi}{3}$ and contains area A_1, while the inner loop is traced out from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ (with negative radius) and contains area A_2. The desired area is then $A = A_1 - A_2$. First, we compute the indefinite integral

$$\int (1+2\cos(\theta))^2\,d\theta = \int (1+4\cos(\theta)+4\cos^2(\theta))\,d\theta = \int (1+4\cos(\theta)+2(1+\cos(2\theta)))\,d\theta$$
$$= \int (3+4\cos(\theta)+2\cos(2\theta))\,d\theta = 3\theta + 4\sin(\theta) + \sin(2\theta).$$

Then we compute the two separate areas (since they are traced out for different intervals)

$$\begin{aligned}
A_1 &= \int_{-2\pi/3}^{2\pi/3} \frac{1}{2} r^2 \, d\theta = 2\int_0^{2\pi/3} \frac{1}{2}(1+2\cos(\theta))^2 \, d\theta \\
&= [3\theta + 4\sin(\theta) + \sin(2\theta)]_0^{2\pi/3} = 2\pi + \frac{3\sqrt{3}}{2} \\
A_2 &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 \, d\theta = 2\int_{2\pi/3}^{\pi} \frac{1}{2}(1+2\cos(\theta))^2 \, d\theta \\
&= [3\theta + 4\sin(\theta) + \sin(2\theta)]_{2\pi/3}^{\pi} = \pi - \frac{3\sqrt{3}}{2} \\
A &= A_1 - A_2 = \pi + 3\sqrt{3}.
\end{aligned}$$

Part B

7. (20 points)

(a) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^6}$$

is absolutely convergent, conditionally convergent, or divergent.

Answer:

The series converges by the alternating series test. It converges absolutely by the intgeral test or the p-test.

(b) Estimate the sum of the series with an accuracy of $.01 = 1/100$.

Answer:

The alternating series is

$$1 - \frac{1}{2^6} + \frac{1}{3^6} + \cdots = 1 - \frac{1}{64} + \frac{1}{729} + \cdots$$

Its third terms is less that $.005 = 1/200$, so the sum of the first two terms will give the desired precision. That sum is

$$1 - \frac{1}{64} = \frac{63}{64} = .984375.$$

8. (20 points)

(a) Find a power series representation centered at -1 as well as the radius and interval of convergence for the function

$$f(x) = \frac{x+1}{x-1}$$

Answer:

Write $f(x)$ as the sum $\frac{a}{1-r}$ of a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$.

$$f(x) = \frac{x+1}{x-1} = \frac{-\frac{1}{2}(x+1)}{1-\left(\frac{x+1}{2}\right)} = \sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)(x+1)\left(\frac{x+1}{2}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{(-1)}{2^n}(x+1)^n$$

This converges if and only if $|r| < 1$, i.e. if and only if

$$\mid r \mid = \frac{\mid x+1 \mid}{2} < 1 \Longleftrightarrow \mid x+1 \mid < 2.$$

It follows that the radius of convergence is 2 and the interval of convergence is $(-3, 1)$.

(b) Write the following integral as a power series in $x+1$. What is the radius of convergence of this power series?

$$\int \frac{x+1}{x-1} dx$$

Answer:

Using term-by-term integration, for $\mid x+1 \mid < 2$ we have

$$\begin{aligned}
\int \frac{x+1}{x-1} dx &= \int \sum_{n=1}^{\infty} \frac{(-1)}{2^n}(x+1)^n dx \\
&= \sum_{n=1}^{\infty} \int \frac{(-1)}{2^n}(x+1)^n dx \\
&= C + \sum_{n=1}^{\infty} \frac{(-1)}{(n+1)2^n}(x+1)^{n+1}
\end{aligned}$$

which has radius of convergence 2.

9. (20 points)

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n - \ln n}$$

Answer:

First, consider the series

$$\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n-\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{n-\ln n}$$

for absolute convergence. Since $n > n - \ln n$ for $n \geq 2$,

$$\frac{1}{n-\ln n} \geq \frac{1}{n}.$$

We also know that the harmonic series $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by the p-series test with $p = 1$. Therefore, it follows from the comparison test that the series diverges.

Now, we consider the series

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n-\ln n}$$

for conditional convergence. It is an alternating series satisfying

$$\lim_{n\to\infty}\frac{1}{n-\ln n} = 0.$$

Since

$$\left(\frac{1}{x-\ln x}\right)' = -\frac{1-1/x}{(x-\ln x)^2} < 0$$

for $x \geq 2$, we know that $\frac{1}{n-\ln n}$ is a decreasing for $n \geq 2$. So by the Alternating Series test, the original series converges.

Therefore, the series is a conditionally convergent series.

10. (20 points)

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n}}.$$

Answer:

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{2^n|x-3|^n}$$
$$= \lim_{n\to\infty} 2\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\cdot|x-3| = 2|x-3|.$$

From

$$2|x-3| < 1 \Leftrightarrow |x-3| < \frac{1}{2},$$

the radius of convergence $R = 1/2$.

Now consider the boundary case $x = 5/2$ or $x = 7/2$. Plugging $x = 5/2$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},$$

which converges by the alternating series test.

Plugging $x = 7/2$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges by the p-series test with $p = 1/2 < 1$.

So the interval of convergence is $[5/2, 7/2)$.

11. (20 points) Let $f(x) = \dfrac{x}{x^2+4}$.

(a) Find a power series expansion for $f(x)$ about $x = 0$. Write it in the form $\sum_{n=0}^{\infty} (-1)^{e_n} a_n x^{p_n}$.

Answer:

$$\frac{x}{x^2+4} = \frac{x/4}{1-(-\frac{x^2}{4})} = \sum_{n=0}^{\infty} \frac{x}{4}\left(\frac{-x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n+1}$$

so $e_n = n$, $a_n = \frac{1}{4^{n+1}}$, and $p_n = 2n+1$.

(b) Find the radius and interval of convergence for the series you found in (a).

Answer:

This is a geometric series with $|r| = |\frac{x^2}{4}|$ so it converges absolutely when $|\frac{x^2}{4}| < 1$ or $|x| < 2$ and diverges otherwise (note that because it's geometric, we do not need to check the endpoints; we know it diverges at both endpoints). Thus, the radius of convergence is $R = 2$, and the interval of convergence is $(-2, 2)$.

(c) Find $f^{(5)}(0)$ and $f^{(10)}(0)$.

Answer:

$f^{(5)}(0) = 5! \cdot c_5 = \frac{5!}{4^3}$ (Note: use n = 2 in the series to find c_5)
$f^{(10)}(0) = 10! \cdot c_{10} = 10! \cdot 0 = 0$ (Note: the series has only odd powers of x, so all even-index coefficients are zero.)

Scratch paper

More scratch paper

And even more scratch paper