MATH 162

Final Exam ANSWERS

December 8, 2005

Part A

1. (30 points)

(a) (10 points) Calculate

$$\int x\sin(x^2)\,dx$$

(b) (10 points) Calculate

$$\int x\ln(x^2)\,dx$$

(c) (10 points) Calculate

$$\int \frac{dy}{y(y^2-1)}$$

Answer:

(a) Use the substitution $y = x^2$, $dy = 2xdx$ to get

$$\begin{aligned}\int x\sin(x^2)\,dx &= \frac{1}{2}\int \sin(y)\,dy \\ &= -\frac{1}{2}\cos(y) + C \\ &= -\frac{1}{2}\cos(x^2) + C\end{aligned}$$

(b) We could use the same substitution as in part (a). On the other hand, $\ln(x^2) = 2\ln(x)$. So, using integration by parts with

$$\begin{aligned} u &= \ln(x) & dv &= x\,dx \\ du &= \frac{1}{x}\,dx & v &= \frac{x^2}{2}\end{aligned}$$

we get

$$
\begin{aligned}
\int x \ln(x^2)\, dx &= 2 \int x \ln(x)\, dx \\
&= 2\ln(x)\frac{x^2}{2} - 2\int \frac{x^2}{2} \cdot \frac{1}{x}\, dx \\
&= x^2 \ln(x) - \int x\, dx \\
&= x^2 \ln(x) - \frac{x^2}{2}
\end{aligned}
$$

(c) We use partial fractions. First,

$$\frac{1}{y(y^2-1)} = \frac{1}{y(y-1)(y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}.$$

Calculating A, B, C, we find

$$1 = A(y-1)(y+1) + By(y+1) + Cy(y-1).$$

Setting $y = 0$, we get $A = -1$; setting $y = 1$ we get $B = 1/2$; setting $y = -1$ we get $C = 1/2$. Therefore,

$$
\begin{aligned}
\int \frac{dy}{y(y^2-1)} &= -\int \frac{dy}{y} + \frac{1}{2}\int \frac{dy}{y-1} + \frac{1}{2}\int \frac{dy}{y+1} \\
&= -\ln|y| + \frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y+1| + C.
\end{aligned}
$$

2. (10 points) Set up a integral which presents the surface area obtained by rotating the curve given by the function below about the line $x = -1$. (First sketch a picture to make sure that you are rotating around the correct line.)

$$y = \frac{1}{3}(x^2+2)^{3/2} \qquad 1 \le x \le 2$$

Answer:

First,

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2}\, 2x = x(x^2+2)^{1/2}.$$

The formula for the surface area of curves rotated about lines parallel to the y-axis gives

$$\begin{aligned}
A &= \int_a^b 2\pi R\sqrt{dx^2+dy^2} \\
&= 2\pi \int_1^2 (x-(-1))\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\pi \int_1^2 (x+1)\sqrt{1+x^2(x^2+2)}\,dx \\
&= 2\pi \int_1^2 (x+1)\sqrt{x^4+2x^2+1}\,dx \\
&= 2\pi \int_1^2 (x+1)(x^2+1)\,dx.
\end{aligned}$$

The integral could also be set up in terms of y.

3. (12 points) A hemispherical reservoir is filled with water. The radius of the hemisphere is R ft. The weight of water is 62.5 lb/ft^3. How much work is required to pump the water out of the reservoir until the height(depth) of the water which is left is $R/2$. (Your answer will be in terms of R.)

Answer:

A slice of water at depth x will have radius $\sqrt{R^2-x^2}$. Its area will be $\pi(R^2-x^2)$. If it has thickness dx, then the weight of the water in that slice will be $62.5\pi(R^2-x^2)dx$. The work required to raise the water in the slice to the top, where $x=0$, will be $62.5\pi x(R^2-x^2)\,dx$. Integrating this amount of work from $x=0$ to $x=R/2$, we get

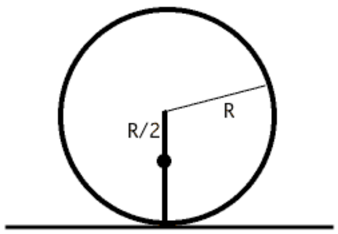
$$\begin{aligned}
\text{Work} &= \int_0^{R/2} 62.5\pi x(R^2-x^2)\,dx \\
&= 62.5\pi \int_0^{R/2} (R^2x-x^3)dx \\
&= 62.5\pi \left(R^2\frac{x^2}{2}-\frac{x^4}{4}\right)\Bigg|_0^{R/2} \\
&= 62.5\pi \left(\frac{R^4}{2^3}-\frac{R^4}{2^6}\right) \\
&= 21.5R^4 \quad \text{ft} - \text{lb}.
\end{aligned}$$

The unit of work in the British system is foot-pounds. Note that in the British system, weight is already in units of force, so you don't have to multiply by g.

4. (14 points) A bicycle wheel of radius R rolls along a flat surface at a rate of v feet per second. There is a reflector half way out toward the rim of the wheel. (see figure).

(a) (7 points) Write equations for the x and y coordinates of the reflector as functions of time t.

(b) (7 points) What is the velocity of the reflector in the x direction when it is at its highest point?



(see figure)

Answer:

(a) The equation for the center is $x = vt$, $y = R$. From the perspective of the center, the reflector has equation $x = -(R/2)\sin(tv/R)$, $y = -(R/2)\cos(tv/R)$. The term $R/2$ appears because the distance of the reflector from the center is $R/2$. The term appears because for a wheel of radius R, the angular velocity should be v/R. There could be other equations, depending on where the reflector starts. The combined equations would be

$$x(t) = vt - \frac{R}{2}\sin\left(\frac{tv}{R}\right)$$
$$y(t) = R - \frac{R}{2}\cos\left(\frac{tv}{R}\right)$$

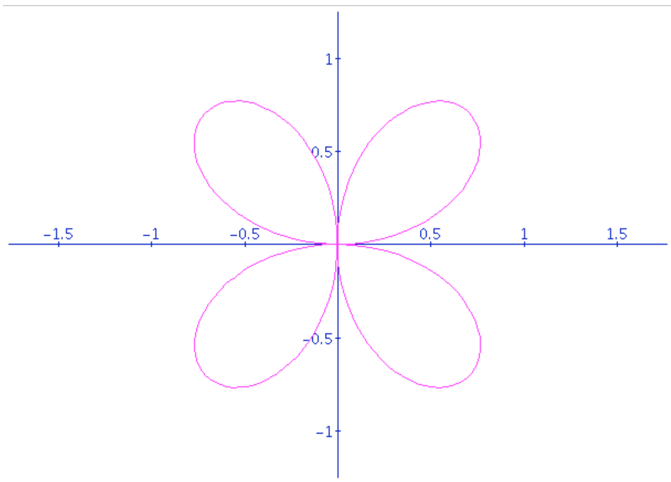
(b) The reflector would be at its highest point when $tv/R = \pi$. We get

$$\frac{dx}{dt} = v - \frac{R}{2}\cdot\frac{v}{R}\cos(\pi) = \frac{3v}{2}.$$

5. (10 points) Make a rough sketch of the graph of $r = \sin(2\theta)$.

Set up an integral which calculates the area of one petal of the graph of $r = \sin(2\theta)$.

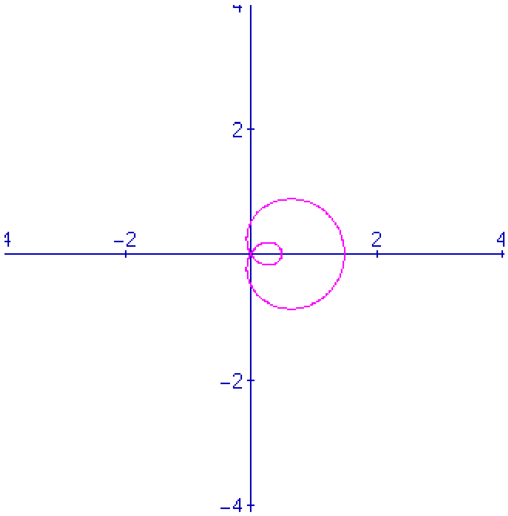
Answer:



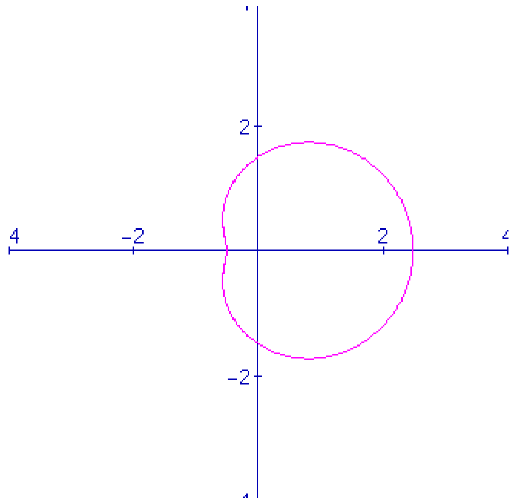
One petal goes from $\theta = 0$ to $\theta = \pi/2$, so the area is

$$A = \int_0^{\pi/2} \frac{1}{2} r^2(\theta) d\theta = \int_0^{\pi/2} \frac{1}{2} \sin^2(2\theta) d\theta.$$

6. (24 points) Match the equations with the graphs. The equations are given on the next page. **Record your answers on the answer sheet, not on this sheet.**



A: ____________________



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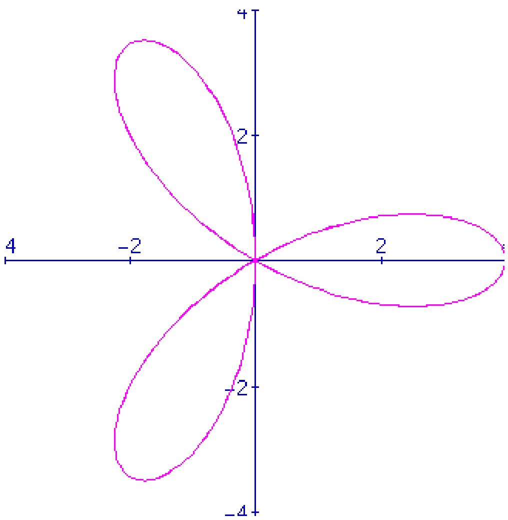

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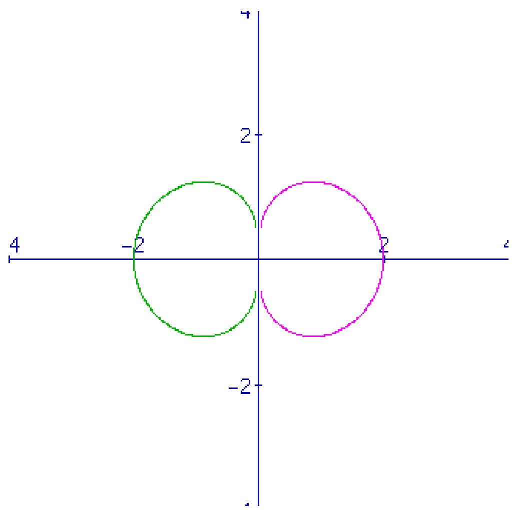

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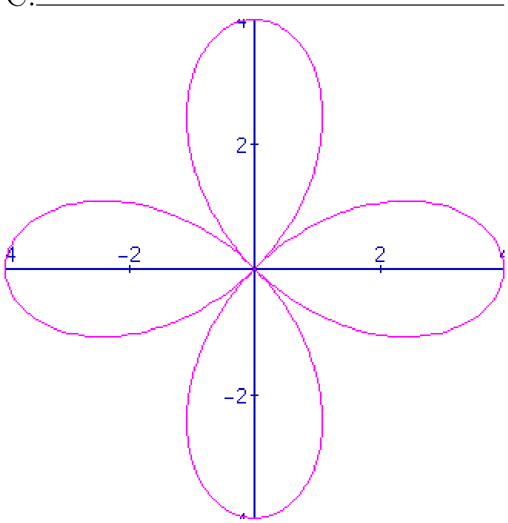
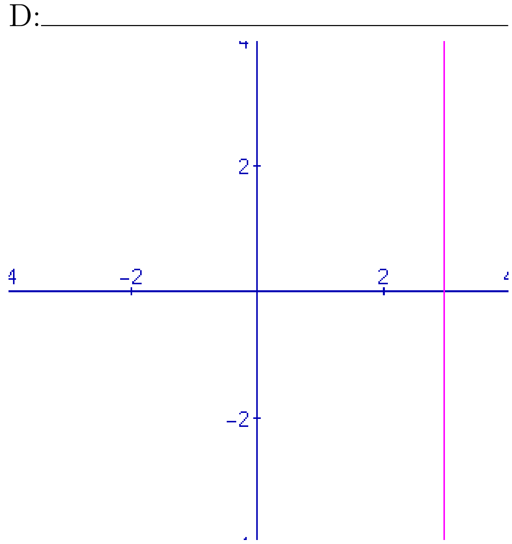


E: ____________________



F: ____________________

1. $r^2 = 4\cos^2\theta + \sin^2\theta$
2. $r^2 = 4\sin^2\theta + \cos^2\theta$
3. $r = 3/\cos\theta$
4. $r = 3/\sin\theta$
5. $r = 1.5 + \cos\theta$
6. $r = 1.0 + \cos\theta$
7. $r = 0.5 + \cos\theta$
8. $r = 4\cos\theta$
9. $r = 4\cos 2\theta$
10. $r = 4\cos 3\theta$
11. $r = 4\cos 4\theta$
12. $r^2 = 4\cos\theta$
13. $r^2 = 4\cos 2\theta$
14. $r^2 = 4\cos 3\theta$

Answer:

A: 7	B: 5
C: 10	D: 1 or 12
E: 9	F: 3

Part B

1. (33 points) (a) (11 points) Is the following series convergent or divergent? You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^2+3}{2n^2+4n+7}$$

(b) (11 points) Is the following series convergent or divergent? You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5n}$$

(c) (11 points) Is the following series convergent or divergent? You must justify your answer.

$$\sum_{n=1}^{\infty} \frac{1}{2^{(n^2)}}$$

Answer:

(a) Since

$$\lim_{n\to\infty} \frac{n^2+3}{2n^2+4n+7} = \frac{1}{2} > 0$$

it follows from the divergence test that the series diverges.

(b) If we just keep the leading terms in the numerator and denominator, we get $n^2/n^3 = 1/n$. But $\sum_{n=1}^{\infty} 1/n$ diverges. This suggests using the limit comparison test. Since

$$\lim_{n\to\infty} \frac{\frac{n^2+4}{n^3+5n}}{\frac{1}{n}} = 1$$

it follows from the limit comparison test that the original series diverges.

(c) Since $\sum_{n=1}^{\infty} 1/2^n$ converges, and since $0 < 1/2^{(n^2)} < 1/2^n$, it follows that the original series converges.

2. (11 points) Is the following series absolutely convergent, conditionally convergent, or divergent? You must justify your answer.

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

Answer:

The terms of the series are decreasing to 0 in absolute value, and the series is an alternating series. Therefore it converges. But

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$$

is a divergent p-series. Therefore, the original series converges conditionally.

3. (11 points) Find the radius of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$$

Answer:

Using the ratio test,

$$\left| \frac{\frac{(n+1)^2}{2^{n+1}} x^{n+1}}{\frac{n^2}{2^n} x^n} \right| = \left| \left(\frac{n+1}{n} \right)^2 \frac{x}{2} \right|$$
$$\to \left| \frac{x}{2} \right| = L$$

as $n \to \infty$. We get convergence when $L < 1$, which gives us $|x| < 2$. Therefore, the radius of convergence is 2.

4. (11 points) Find the Taylor polynomial

$$T_2(x) = c_0 + c_1 x + c_2 x^2$$

for

$$f(x) = \sqrt{1 + e^x}.$$

Answer:

n	0	1	2
$f^{(n)}(x)$	$(1+e^x)^{1/2}$	$\frac{1}{2}e^x(1+e^x)^{-1/2}$	$\frac{1}{2}e^x(1+e^x)^{-1/2} - \frac{1}{4}e^{2x}(1+e^x)^{-3/2}$
$f^{(n)}(0)$	$2^{1/2}$	$2^{-3/2}$	$2^{-3/2} - 2^{-7/2} = 3 \cdot 2^{-7/2}$

Therefore,

$$\begin{aligned} T_2(x) &= 2^{1/2} + 2^{-3/2}x + \frac{3}{2!} \cdot 2^{-7/2}x^2 \\ &= 2^{1/2} + 2^{-3/2}x + 3 \cdot 2^{-9/2}x^2. \end{aligned}$$

5. (12 points) Suppose that

$$f^{(5)}(x) = \frac{\cos^2 x}{e^x + 3}.$$

(a) (6 points) Find a number M such that

$$|f^{(5)}(x)| \leq M$$

for all x.

(b) (6 points) For which x can you guarantee that the error $|R_4(x)|$ is less than 10^{-7}?

Find a bound for the remainder term $|R_4(x)|$, valid for all values of x.

Answer:

(a) Since $|\cos^2 x| \leq 1$ and $1/(e^x + 3) \leq 1/3$, we can choose $M = 1/3$.

(b) By the remainder theorem,

$$|R_4(x)| \leq \frac{1}{3} \cdot \frac{|x|^5}{5!} = \frac{|x|^5}{360}$$

Since we want the remainder to be less than 10^{-7}, we must solve

$$\frac{|x|^5}{360} < 10^{-7}$$

or

$$|x| < \left(360 \times 10^{-7}\right)^{1/5} = \left(3.60 \times 10^{-5}\right)^{1/5} = .129$$

6. (11 points) Find the Taylor series $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$ for

$$f(x) = xe^{x^3}.$$

(Write out at least the first three non-zero terms.)

Answer:

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

we substitute x^3 in place of x to get

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \ldots$$

and finally, multiplying by x, we get

$$xe^{x^3} = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + \ldots$$

7. (11 points) The Taylor series for $f(x) = x\sin(x^2)$ is

$$f(x) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \ldots$$

Find $f^{(11)}(0)$.

Answer:

In a Taylor series, the coefficient of x^{11} is $f^{(11)}(0)/11!$. In the above series, the coefficient is $1/5!$. Therefore, $f^{(11)}(0)/11! = 1/5!$. Thus,

$$f^{(11)}(0) = \frac{11!}{5!}$$