

# Math 162: Calculus IIA

## Final Exam ANSWERS

December 17, 2010

### Part A

#### 1. (20 points)

(a) Find the partial fraction expansion of

$$\frac{1}{x^3 - x}.$$

(b) Calculate the integral

$$\int \frac{1}{x^3 - x} dx.$$

NOTE: The first part of this problem was designed to help you do the second part. If you did the first part incorrectly, you will not get partial credit for “correctly” using the wrong partial fraction expansion to find the integral.

**Solution:** (a)  $x^3 - x = (x + 1)x(x - 1)$ . This means that

$$\frac{1}{x^3 - x} = \frac{A}{x + 1} + \frac{B}{x} + \frac{C}{x - 1}.$$

Putting the right hand side on a common denominator we get

$$\begin{aligned} \frac{1}{x^3 - x} &= \frac{A(x^2 - x) + B(x^2 - 1) + C(x^2 + x)}{x^3 - x} \\ &= \frac{(A + B + C)x^2 + (A - C)x - B}{x^3 - x} \end{aligned}$$

which gives  $A = C = 1/2$  and  $B = -1$ . so

$$\frac{1}{x^3 - x} = \frac{1}{2(x + 1)} - \frac{1}{x} + \frac{1}{2(x - 1)}$$

**Solution:** (b) From (a) we get

$$\int \frac{1}{x^3 - x} dx = \int \left( \frac{1}{2(x + 1)} - \frac{1}{x} + \frac{1}{2(x - 1)} \right) dx.$$

Now

$$\begin{aligned}\int \frac{1}{2(x+1)} dx &= \frac{1}{2} \ln|x+1| = \ln\sqrt{|x+1|} \\ -\int \frac{1}{x} dx &= -\ln|x| = \ln(1/|x|) \\ \int \frac{1}{2(x-1)} dx &= \frac{1}{2} \ln|x-1| = \ln\sqrt{|x-1|}\end{aligned}$$

so

$$\begin{aligned}\int \frac{1}{x^3-x} dx &= \ln\sqrt{|x+1|} + \ln(1/|x|) + \ln\sqrt{|x-1|} + c \\ &= \ln\left|\frac{\sqrt{x^2-1}}{x}\right| + c.\end{aligned}$$

## 2. (20 points)

Evaluate the integral

$$\int \frac{x+1}{\sqrt{x^2+4}} dx$$

**Solution:** A right triangle with sides 2 and  $x$  will have hypotenuse  $\sqrt{x^2+4}$ . Let  $\theta$  be the angle opposite the side with length  $x$ . Then we have

$$\begin{aligned}x &= 2 \tan \theta \\ dx &= 2 \sec^2 \theta d\theta \\ \sqrt{x^2+4} &= 2 \sec \theta\end{aligned}$$

This gives

$$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+4}} dx &= \int \frac{(2 \tan \theta + 1)2 \sec^2 \theta}{2 \sec \theta} d\theta \\ &= 2 \int \tan \theta \sec \theta d\theta + \int \sec \theta d\theta \\ &= 2 \int \frac{\sin \theta}{\cos^2 \theta} d\theta + \int \sec \theta d\theta\end{aligned}$$

The second integral is  $\ln|\sec \theta + \tan \theta|$ . For the first we substitute  $u = \cos \theta$  and get

$$2 \int \frac{\sin \theta}{\cos^2 \theta} d\theta = -2 \int \frac{du}{u^2} = \frac{2}{u} = 2 \sec \theta.$$

Hence

$$\begin{aligned} \int \frac{x+1}{\sqrt{x^2+4}} dx &= 2 \int \frac{\sin \theta}{\cos^2 \theta} d\theta + \int \sec \theta d\theta \\ &= 2 \sec \theta + \ln |2 \sec \theta + 2 \tan \theta| + c \\ &= \sqrt{x^2+4} + \ln |\sqrt{x^2+4} + x| + c. \end{aligned}$$

### 3. (15 points)

Rotate the region bounded by  $y = 0$ ,  $y = \sin x$ ,  $x = 0$  and  $x = \pi$  around the  $x$ -axis. Compute the volume of the resulting body.

**Solution:**

$$\begin{aligned} V &= \pi \int_0^\pi \sin^2 x dx \\ &= \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx \\ &= \frac{\pi}{2} \left( \pi - \frac{1}{2} [\sin 2x]_0^\pi \right) \\ &= \frac{\pi^2}{2} \end{aligned}$$

### 4. (15 points)

An inverted cone has radius of the base  $2m$  and depth/height  $5m$ . The cone is filled with water up to the height of  $3m$ . How much work (in Joules) is required to empty the cone? The density of water is  $1000kg/m^3$  and  $g = 9.8m/sec^2$ . You may assume  $9.8 \cdot \pi = 31$ . A Joule is the metric unit of work,  $1J = 1kg \cdot m^2/sec^2$

**Solution:** The work required to lift a slice of water from height  $x$  to the top is:

$$dW = \text{density} \cdot g \cdot (5 - x) \cdot dV$$

$$dV = \pi(\text{radius})^2 dx = \pi \left( \frac{2}{5} \right)^2 x^2 dx$$

The total work is then

$$\begin{aligned}
W &= 1000g\pi \frac{4}{25} \int_0^3 (5-x)x^2 dx \\
&= 160g\pi \left( \frac{5}{3}3^3 - \frac{1}{4}3^4 \right) \\
&= 160g\pi \frac{99}{4} = 40 \cdot 31 \cdot 99J = 122760J
\end{aligned}$$

**5. (20 points)**

The cardioid is the curve defined in polar coordinates by  $r = 1 + \cos \theta$ . Find the arclength of the cardioid for  $0 \leq \theta \leq \pi$ . You may use the identity  $\cos^2(\theta/2) = (1 + \cos \theta)/2$ .

**Solution:** The arclength formula in polar coordinates is (for  $\alpha \leq \theta \leq \beta$ )

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

In our case

$$\begin{aligned}
\frac{dr}{d\theta} &= -\sin \theta \\
r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 + \cos \theta)^2 + \sin^2 \theta \\
&= 1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta \\
&= 2 + 2\cos \theta \\
&= 4\cos^2(\theta/2) \\
\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} &= |2\cos(\theta/2)|
\end{aligned}$$

so our arclength is

$$\begin{aligned}
s &= \int_0^{\pi} |2\cos(\theta/2)| d\theta \\
&= 4 \int_0^{\pi/2} |\cos u| du \quad \text{where } u = \theta/2 \\
&= [4\sin u]_0^{\pi/2} \\
&= 4.
\end{aligned}$$

**6. (10 points)**

Find the area of the region bounded by  $y = \sin(x)$  and  $y = \cos(x)$  for  $\pi/4 \leq x \leq 5\pi/4$ .

**Solution:** First we note that  $\sin(x) \geq \cos(x)$  for  $0 \leq x \leq \pi/4$ . We get

$$\begin{aligned} A &= \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x) \, dx \\ &= [-\cos(x) - \sin(x)]_{\pi/4}^{5\pi/4} \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ &= 2\sqrt{2}. \end{aligned}$$

**Part B****7. (20 points)**

(a) Find the Maclaurin series expansion of  $\sin x/x$ , as well as the interval of convergence.

(b) Find the Maclaurin series for

$$\int_0^x \frac{\sin t}{t} dt,$$

as well as the interval of convergence.

**Solution:** (a) The series for  $\sin x$  is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$

which converges for all  $x$ . Therefore the series for  $\sin x/x$  is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots,$$

which also converges for all  $x$ .

**Solution:** (b) We can find this series by integrating the one above. The result is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \cdots,$$

which again converges for all  $x$ .

**8. (20 points)**

(a) Find the Taylor series centered at 0 of the function  $\sin(x^2)$ , as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x \sin(t^2) dt$$

as a power series in  $x$ .

**Solution:** (a). The Taylor series of  $\sin(x)$  is

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$

for all  $x$ . Therefore

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots$$

for all  $x$ .

(b).

$$\begin{aligned} \int_0^x \sin(t^2) dt &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{4n+2} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3} x^{4n+3} \\ &= \frac{x^3}{3} - \frac{x^7}{3! \cdot 7} + \frac{x^{11}}{5! \cdot 11} + \cdots \end{aligned}$$

The equation holds for all  $x$ .

**9. (20 points)** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln(n)}.$$

**Solution:** Using ratio test, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{4^{n+1} \ln(n+1)} \frac{4^n \ln(n)}{x^n} \right| \\ &= \left| \frac{x}{4} \right| \cdot \left| \lim_{n \rightarrow \infty} \frac{\ln(n)}{\ln(n+1)} \right| \\ &= \left| \frac{x}{4} \right| \cdot \left| \lim_{n \rightarrow \infty} \frac{1/n}{1/(n+1)} \right| \\ &= \left| \frac{x}{4} \right| \end{aligned}$$

To have an absolute convergence series, we need to have  $\left| \frac{x}{4} \right| < 1$ , so  $|x| < 4$  and the radius of convergence is 4. Consider the end points at  $x = 4$  and  $x = -4$ , we have

- $x = 4$ : The series is equal to

$$\sum_{n=2}^{\infty} (-1)^n \frac{4^n}{4^n \ln(n)} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$$

Since  $1/\ln(n)$  is decreasing and converges to zero, by alternating series test, the above series is convergent.

- $x = -4$ : The series is equal to

$$\sum_{n=2}^{\infty} (-1)^n \frac{(-4)^n}{4^n \ln(n)} = \sum_{n=2}^{\infty} (-1)^n \frac{(-1)^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

Since  $0 < \ln(n) < n$  for  $n \geq 2$ , the above series is larger than the harmonic series  $\sum_{n=2}^{\infty} \frac{1}{n}$ , which is divergent. By comparison test, the above series is divergent.

Therefore the interval of convergence of the series is  $(-4, 4]$ .

## 10. (20 points)

Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^2 + 1}}{n^2}$$

is absolutely convergent, conditionally convergent or divergent.

**Solution:** The series is alternating. So let  $a_n = \sqrt{n^2 + 1}/n^2$ , then

$$a_n = \sqrt{\frac{n^2 + 1}{n^4}} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4}}$$

and it is clear that  $a_n$  is decreasing as  $n$  increases. Also  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . By alternating series test,  $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^2+1}}{n^2}$  is convergent.

Now consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\sqrt{n^2+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2} = \sum_{n=1}^{\infty} a_n$$

As

$$\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+1}/n^2}{1/n} = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = 1$$

and  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent,  $\sum_{n=1}^{\infty} a_n$  is divergent. Thus  $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^2+1}}{n^2}$  is conditionally convergent.



**11. (20 points)**

(a) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$$

is absolutely convergent, conditionally convergent or divergent.

(b) Estimate the sum of the series within an accuracy of  $\frac{1}{16}$ .

**Solution:** (a) By root test, let  $a_n = (-2)^n/n^n$ , then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0 < 1$$

Therefore this series is absolute convergent.

(b) Let  $S$  be the sum of the series  $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} (-1)^n b_n$ , where  $b_n = 2^n/n^n$ , and let  $S_n$  be the  $n$ th partial sum. The alternating series estimate says that  $|S - S_n| < b_{n+1}$ . Therefore we need to find the  $n$  such that  $b_{n+1} \leq \frac{1}{16}$  which guarantees the error is smaller than  $\frac{1}{16}$ .

$$2^{n+1}/(n+1)^{n+1} = (2/(n+1))^{n+1} \leq \frac{1}{16}$$

implies that  $n$  is at least 3. Thus to approximate the sum, we use

$$-2 + 1 - \frac{8}{27} = -\frac{35}{27}.$$