# Math 162: Calculus IIA

## Final Exam ANSWERS

December 17, 2009

**Part A**

**1. (20 points)**

(a) Find the partial fraction expansion of

$$\frac{-3x-2}{x^3-x^2+4x-4}.$$

(b) Calculate the integral

$$\int \frac{-3x-2}{x^3-x^2+4x-4}\,dx.$$

NOTE: The first part of this problem was designed to help you do the second part. If you did the first part incorrectly, you will not get partial credit for "correctly" using the wrong partial fraction expansion to find the integral.

**Solution:** (a) The only real root of the denominator is 1, and from polynomial division we get $x^3-x^2+4x-4=(x-1)(x^2+4)$. This means that

$$\frac{-3x-2}{x^3-x^2+4x-4}=\frac{A}{(x-1)}+\frac{Bx+C}{x^2+4}$$

Putting the right hand side on a common denominator we get

$$\frac{-3x-2}{x^3-x^2+4x-4}=\frac{(A+B)x^2+(C-B)x+4A-C}{x^3-x^2+4x-4}$$

and equating coefficients yields the equations $A+B=0$, $C-B=-3$ and $4A-C=-2$, which have the solution $A=-1$, $B=1$ and $C=-2$.

**Solution:** (b) From (a) we get

$$\int \frac{-3x-2}{x^3-x^2+4x-4}\,dx=\int \frac{-1}{(x-1)}+\frac{x-2}{x^2+4}\,dx=\int \frac{-1}{(x-1)}+\frac{x}{x^2+4}-\frac{2}{x^2+4}\,dx.$$

We have $\int \frac{-1}{(x-1)}\,dx=-\ln|x-1|+C_1$. By substituting $u=x^2+4$ we have $\int \frac{x}{x^2+4}\,dx=\frac{1}{2}\ln|x^2+4|+C_2$. Finally we substitute $x=2\tan(\theta)$ and $dx=2\sec^2(\theta)d\theta$ in the last integral to get

$$\int \frac{-2}{x^2+4}\,dx=\int -2\frac{2\sec^2(\theta)}{4\sec^2(\theta)}\,d\theta=-\arctan(\frac{x}{2})+C_3.$$

Combining this we get

$$\int \frac{-3x-2}{x^3-x^2+4x-4}\,dx = -\ln|x-1| + \frac{1}{2}\ln|x^2+4| - \arctan(\frac{x}{2}) + C.$$

**2. (20 points)**

Evaluate the integral

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

**Solution:** We let $x = 2\sin(\theta)$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 2\cos(\theta)\,d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)} = \sqrt{4\cos^2(\theta)} = 2|\cos(\theta)| = 2\cos(\theta)$$

since $\cos(\theta) \geq 0$ on the interval $-\pi/2 \leq \theta \leq \pi/2$. This means that

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos(\theta)}{4\sin^2(\theta)} 2\cos(\theta)\,d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\,d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C.
\end{aligned}$$

Consider $\theta$ as the angle in a right triangle, then $\sin(\theta) = \frac{x}{2}$, and so we label the opposite side and hypotenuse as having lengths $x$ and 2. From the Pythagorean theorem we get

$$\cot(\theta) = \frac{\sqrt{4-x^2}}{x}.$$

The result is then

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \arcsin\left(\frac{x}{2}\right) + C.$$

**3. (15 points)**

Rotate the region bounded by $y = 0$, $y = \mathrm{e}^{-x^2}$, $x = 0$ and $x = 1$ around the $y$-axis. Compute the volume of the resulting body.

**Solution:** This is best done with shells. The height is $\mathrm{e}^{-x^2}$, the circumference is $2\pi x$ and the thickness is $dx$. So we get

$$V = \int_0^1 2\pi x\,\mathrm{e}^{-x^2}\ dx = 2\pi \int_0^1 x\,\mathrm{e}^{-x^2}\ dx$$

To calculate this we substitute $u = -x^2$ and get $du = \frac{-\,dx}{2x}$. Hence

$$\begin{aligned} V &= 2\pi \int_0^1 x\,\mathrm{e}^{-x^2}\ dx \\ &= 2\pi \int_0^{-1} e^u \frac{-du}{2} \\ &= \pi \int_{-1}^0 e^u\ du \\ &= \pi[e^u]_{-1}^0 \\ &= \pi(1 - \frac{1}{e}). \end{aligned}$$

**4. (15 points)**

A cylindrical well has radius $1m$ and depth $5m$. The depth of the water in the well is $3m$. How much work (in Joules) is required to empty the well? The density of water is $1000kg/m^3$ and $g = 9.8m/sec^2$. You may assume $9.8 \cdot \pi = 31$. A Joule is the metric unit of work, $1J = 1kg \cdot m^2/sec^2$

**Solution:** The work require to pump out a layer of water of thickness $\Delta x$ at height $x$ below the surface of the water is $F \cdot d = volume \cdot density \cdot g \cdot (x+2)$. The volume of a layer of water of thickness $\Delta x$ is $\pi \cdot \Delta x$ and so $F \cdot d = \pi \cdot \Delta x \cdot 1000 \cdot 9.8 \cdot (x+2)$. The total work required is then

$$\begin{aligned} W &= 1000 \cdot 9.8\pi \int_0^3 x + 2 \, dx \\ &= 31000 \left[ \frac{1}{2}x^2 + 2x \right]_0^3 \\ &= 325500J. \end{aligned}$$

**5. (10 points)**

Find the area of the region bounded by $y = \sin(x)$ and $y = \cos(x)$ for $0 \leq x \leq \pi/4$.

**Solution:** First we note that $\cos(x) \geq \sin(x)$ for $0 \leq x \leq \pi/4$. We get

$$\begin{aligned} A &= \int_0^{\pi/4} \cos(x) - \sin(x)\, dx \\ &= [\sin(x) + \cos(x)]_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \\ &= \sqrt{2} - 1. \end{aligned}$$

**6. (20 points)**

The cycloid is the curve defined by $x(t) = r(t - \sin(t))$ and $y(t) = r(1 - \cos(t))$, where $r$ is a constant. Find the arclength of the cycloid for $0 \le t \le 2\pi$. You may use the identity $\sin^2(t) = (1 - \cos(2t))/2$.

**Solution:** We first note that $dx/dt = r(1 - \cos t)$ and $dy/dt = r \sin t$. The arclength is given by

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2(1-\cos t)^2 + r^2(t/2)\sin t}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos t + \cos^2(t) + \sin^2 t)}\, dt \\
&= r\int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt \\
&= r\int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt \\
&= r\int_0^{2\pi} 2|\sin(t/2)|\, dt \\
&= r\int_0^{2\pi} 2\sin(t/2)\, dt \\
&= 2r\int_0^{2\pi} \sin(t/2)\, dt \\
&= 2r\left[-2\cos(t/2)\right]_0^{2\pi} \\
&= 8r.
\end{aligned}
$$

We used the fact that $\sin t \ge 0$ on $0 \le t \le \pi$ so $\sin(t/2) \ge 0$ on $0 \le t \le 2\pi$ in the seventh equality.

**Part B**

**7. (25 points)**

(a) Find the power series expansion of $1/(1+x^2)$, as well as radius and interval of convergence.

(b) Find the power series for $\arctan(x)$, as well as the radius and interval of convergence.

**Solution:** (a) We rewrite $1/(1+x^2) = 1/(1-(-x^2))$, which we recognize as the sum of the geometric series

$$\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

The series converges for $|-x^2| < 1$, i.e. $|x| < 1$.

**Solution:** (b) We note that

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

so to find the series for $\arctan(x)$ we integrate the series from (a). This is done term-wise

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} + C.\end{aligned}$$

Since $\arctan(0) = 0$, we see that $C = 0$, so we get

$$\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

By Theorem 2 page 730 the radius of convergence is 1, so we know that the series converges for all $x$ in $(-1,1)$. At the point $x = 1$ we see that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges by the alternating series test. At the point $x = -1$ we have to see if the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(-1)^{2n+1}$ converges. The series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(-1)^{2n+1}$ can be written as $-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, which converges by the alternating series test. The interval of convergence is hence $[-1,1]$.

**8. (25 points)**

(a) Find the Taylor series centered at 0 of the function $\mathrm{e}^{-x^2}$, as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x \mathrm{e}^{-t^2}\ dt$$

as a power series.

**Solution:** (a) The Taylor series for $\mathrm{e}^x$ is

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and so we get

$$\begin{aligned} \mathrm{e}^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \end{aligned}$$

Since the series for $\mathrm{e}^x$ is convergent for all $x$, so is the series for $\mathrm{e}^{-x^2}$.

**Solution:** (b) We use the series from (a)

$$\begin{aligned} \int_0^x \mathrm{e}^{-t^2}\ dt &= \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\ dt \\ &= \left[ \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)n!} \right]_0^x \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}. \end{aligned}$$

This expression is valid for all $x > 0$, since the series for $\int \mathrm{e}^{-x^2}\ dx$ is convergent for all $x$.

**9. (15 points)**

Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x+2)^n}{n+1}.$$

**Solution:** We apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x+2)^{n+1}}{n+2} \frac{n+1}{(-1)^n(x+2)^n}\right| = |x+2|\left(\frac{n+1}{n+2}\right).$$

Hence we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+2|.$$

This means that the series converges for $|x+2| < 1$ and hence that the radius of convergence is 1. We then know that the series converges for $-3 < x < -1$, and it remains to check the endpoints. For $x = -3$ we get

$$\sum_{n=0}^{\infty}(-1)^n \frac{(-3+2)^n}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

which is a divergent harmonic series. For $x = -1$ we get

$$\sum_{n=0}^{\infty}(-1)^n \frac{(-1+2)^n}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}$$

which is convergent by the alternating series test. The interval of convergence is then $-3 < x \leq -1$.

**10. (15 points)**

Determine whether the series

$$\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln(n)}$$

is absolutely convergent, conditionally convergent or divergent.

**Solution:** The function $f(x) = \frac{1}{x\ln(x)}$, has $f'(x) = \frac{-\ln(x)-1}{(x\ln(x))^2}$, which is negative for $x \geq 2$, so $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln(n)}$ and $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$, by the monotone convergence theorem, since the sequence is monotone decreasing and bounded by 0 from below (all terms are positive). The series is convergent by the alternating series test.

To see whether the series is absolutely convergent, we use the integral test. The series is absolutely convergent if the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is convergent. By the integral test this is the case if and only if the improper integral

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$$

is convergent.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x\ln(x)}\,dx &= \int_{\ln(2)}^{\infty} \frac{x\,du}{xu} \\
&= \int_{\ln(2)}^{\infty} \frac{1}{u}\,du \\
&= \lim_{t\to\infty} \ln(t) - \ln(2) \\
&= \infty.
\end{aligned}$$

where we put $u = \ln(x)$ and $dx = x\,du$.

**11. (20 points)**

(a) Determine whether the series

$$\sum_{n=0}^{\infty} (-1)^n \, \mathrm{e}^{-n}$$

is absolutely convergent, conditionally convergent or divergent.

(b) Estimate the sum of the series within an accuracy of $\mathrm{e}^{-5}$. You may leave you answer in terms of powers of $e$.

**Solution:** (a) We apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\,\mathrm{e}^{-n-1}}{(-1)^n e^{-n}}\right| = \frac{\mathrm{e}^{-n-1}}{e^{-n}} = e^{-1}.$$

Hence $\lim_{n\to\infty} = \left|\frac{a_{n+1}}{a_n}\right| = \mathrm{e}^{-1} < 1$, so the sequence is absolutely convergent.

**Solution:** (b) Let $s$ be the sum of the series $\sum_{n=0}^{\infty}(-1)^n b_n$ and let $s_n$ be the $n$'th partial sum. The alternating series estimate says that $|s - s_n| < b_{n+1}$. In our case $b_n = \mathrm{e}^{-n}$ and we get that $|s - s_4| < \mathrm{e}^{-5}$, i.e the difference between $s_4$ and the value of the sum is less than $\mathrm{e}^{-5}$. It remains to calculate the sum $s_4$: $s_4 = 1 - \mathrm{e}^{-1} + \mathrm{e}^{-2} - \mathrm{e}^{-3} + \mathrm{e}^{-4}$.