Math 162: Calculus IIA

Final Exam ANSWERS

May 9, 2007

Part I
1. (10 points)

(a) Find the area enclosed by the curves $y = x + 2$ and $y = x^2$.

(b) Find the volume of the solid obtained by rotating this same region about the x-axis.

Answer:

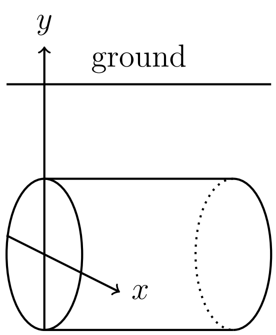
(a) First we need to find where these curves intersect. Setting $x + 2 = x^2$ gives $x^2 - x - 2 = (x-2)(x+1) = 0$, so the curves intersect at $x = 2$ and $x = -1$, i.e., at the points $(2, 4)$ and $(-1, 1)$. So the area is given by:

$$\begin{aligned} A &= \textstyle\int_{-1}^{2} (x + 2 - x^2)\, dx \\ &= (x^2/2 + 2x - x^3/3)\big|_{-1}^{2} \\ &= (2 + 4 - 8/3) - (1/2 - 2 + 1/3) \\ &= 9/2 \end{aligned}$$

(b) When we rotate about the x-axis, a typical vertical cross section is a washer with outer radius $x + 2$ and inner radius x^2. Thus the area is given by $A(x) = \pi[(x+2)^2 - (x^2)^2] = \pi(x^2 + 4x + 4 - x^4)$. So the volume is

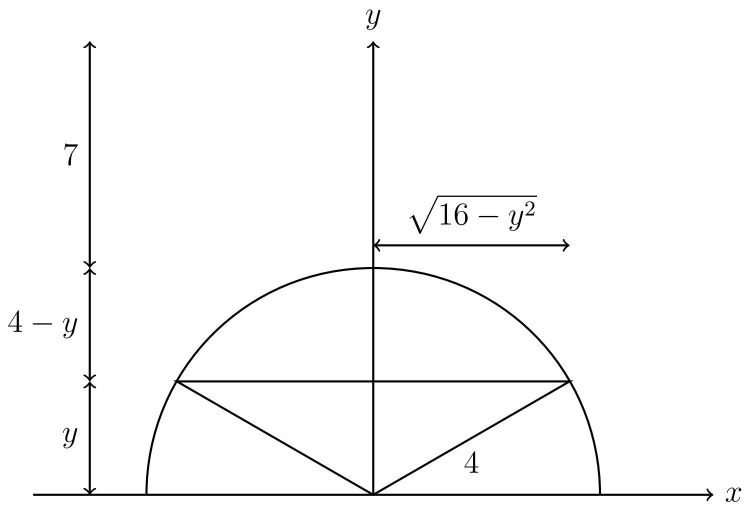
$$
\begin{aligned}
V &= \int_{-1}^{2} A(x)\,dx \\
&= \pi \int_{-1}^{2} (x^2 + 4x + 4 - x^4)\,dx \\
&= \pi(x^3/3 + 2x^2 + 4x - x^5/5)|_{-1}^{2} \\
&= \pi[(8/3 + 8 + 8 - 32/5) - (-1/3 + 2 - 4 + 1/5)] \\
&= 72\pi/5
\end{aligned}
$$

2. (10 points) Gasoline at a service station is stored in a cylindrical tank buried on its side, with the highest part of the tank 5 ft below the surface. The tank is 8 feet in diameter and 10 ft long. The density of gasoline is 45 lb/ft^3. Assume that the filler cap of each automobile is 2 feet above the ground. If the tank is initially full, how much work is done pumping half of the gasoline in the tank into automobiles? (You *do not need to multiply out your answer*, but it should be simplified otherwise.)



Answer:

Consider the following picture of the cross-section of the top half of the tank.



With the choices made in the picture, $0 \le y \le 4$. The volume of the horizontal slice of gasoline in the tank at height y is approximately

$$(length)(width)(thickness) = (10)(2\sqrt{16-y^2})(\Delta y)$$

and thus the amount of work done pumping half of the gasoline out of the tank is

$$\begin{aligned}
W &= \int_0^4 (20\sqrt{16-y^2})(45)(7+(4-y))\,dy \\
&= 20\cdot 45 \int_0^4 ((11-y)\sqrt{16-y^2})\,dy \\
&= 20\cdot 45\cdot 11 \int_0^4 \sqrt{16-y^2}\,dy + 20\cdot 45\cdot \frac{1}{2}\int_0^4 -2y\sqrt{16-y^2}\,dy \\
& \quad (\text{Let } u = 16-y^2,\ du = -2y) \\
&= 20\cdot 45\cdot 11\frac{\pi\cdot 4^2}{4} + 10\cdot 45\int_{16}^0 \sqrt{u}\,du \\
&= 20\cdot 45\cdot 11\pi\cdot 4 + 10\cdot 45\left[\frac{2}{3}u^{3/2}\right]_{16}^0 \\
&= 20\cdot 45\cdot 11\pi\cdot 4 - 10\cdot 45\cdot\frac{2}{3}(16)^{3/2} \\
&= 39,600\pi - 19,200 \approx 105,207.069082 \text{ ft-lb}
\end{aligned}$$

3. (10 points) Find the definite integral

$$\int_0^{2\pi} x\sin x\,dx$$

Answer:

We use integration by parts with

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

so

$$\begin{aligned} \int_0^{2\pi} x\sin x\,dx &= -x\cos x|_0^{2\pi} + \int_0^{2\pi} \cos x\,dx \\ &= -2\pi \end{aligned}$$

4. (10 points) Solve this integral:

$$\int \frac{x^2}{(\sqrt{16-x^2})^3}\,dx$$

Answer:

We use the substitution $x = 4\sin t$, so that $dx = 4\cos t \cdot dt$ and $\sqrt{16-x^2} = 4\cos t$. Then

$$\begin{aligned} \int \frac{x^2}{(\sqrt{16-x^2})^3}\,dx &= \int \frac{16\sin^2 t \cdot 4\cos t}{64\cos^3 t}\,dt \\ &= \int \tan^2 t\,dt \\ &= \int (\sec^2 t - 1)\,dt \\ &= \tan t - t + C \end{aligned}$$

Drawing a triangle, we see that $\tan t - t + C$ reduces to

$$\frac{x}{\sqrt{16-x^2}} - \arcsin(x/4) + C$$

5. (10 points) Evaluate this integral:

$$\int_{-3}^{-1} \frac{1}{x(2x+1)}\,dx$$

Answer:

We use partial fractions:

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

Then bringing to a common denominator,

$$1 = A(2x+1) + Bx,$$

and it follows that $A = 1$ an $B = -2$. So the integral becomes

$$\begin{aligned}\int_{-3}^{-1} \left(\frac{1}{x} - \frac{2}{2x+1}\right) dx &= \left(\ln|x| - \ln|2x+1|\right)\Big|_{-3}^{-1} \\ &= \ln 5 - \ln 3\end{aligned}$$

6. (10 points)

Does the following series converge or diverge? Why or why not?

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

Answer:

Notice this is an alternating series. Let $b_n = \frac{\ln(n)}{n}$. The b_n are nonnegative. Let $f(x) = \frac{\ln(x)}{x}$. Since

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2} < 0$$

if and only if $1 < \ln(x)$, which is equivalent to $e < x$, the sequence $\{b_n\}$ is decreasing for $e < 3 \leq n$. Since

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty}$$

we may apply L'Hospital's rule to obtain

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \stackrel{L'Hop}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

and so the sequence $\{b_n\}$ is contracting to zero. Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the alternating series test.

7. (10 points)

(a) Find the limit $\displaystyle\lim_{n\to\infty} \frac{2n^2 + n - 3}{5n^2 + 2}$.

(b) Does the series $\sum_{n=1}^{\infty} \frac{2n^2+n-3}{5n^2+2}$ converge or diverge?

Answer:

(a)

$$\begin{aligned} \lim_{n\to\infty} \frac{2n^2+n-3}{5n^2+2} &= \lim_{n\to\infty} \frac{2+1/n-3/n^2}{5+2/n^2} \\ &= 2/5 \end{aligned}$$

(b) The general term of the series is $a_n = \frac{2n^2+n-3}{5n^2+2}$. In part (a), we showed $\lim_{n\to\infty} a_n \neq 0$. Thus, the series diverges by the Divergence Test.

8. (10 points) Evaluate $\int_1^{\infty} \frac{x^3}{(x^4+1)^{10}}\, dx$.

Answer:

Let $u = x^4+1$ so that $du = 4x^3 dx$ and $du/4 = x^3 dx$. The integral becomes:

$$\begin{aligned} \int_1^{\infty} \frac{x^3}{(x^4+1)^{10}}\, dx &= \frac{1}{4}\int_2^{\infty} \frac{du}{u^{10}} \\ &= \lim_{t\to\infty} \frac{1}{4}\int_2^{t} \frac{du}{u^{10}} \\ &= \lim_{t\to\infty} \frac{-1}{36}\left(\frac{1}{u^9}\right)\Big|_2^t \\ &= \lim_{t\to\infty} \frac{-1}{36}\left(\frac{1}{t^9} - \frac{1}{512}\right) \\ &= \tfrac{1}{18432} \end{aligned}$$

9. (10 points) Does the series $\sum_{n=1}^{\infty} \frac{n^3}{(n^4+1)^{10}}$ converge or diverge?

Answer:

Notice that $f(x) = \frac{x^3}{(x^4+1)^{10}}$ is continuous and positive on the interval $[1,\infty)$. Also, since

$$
\begin{aligned}
f'(x) &= \frac{3x^2(x^4+1)^{10} - 40x^6(x^4+1)^9}{(x^4+1)^{20}} \\
&= \frac{x^2(x^4+1)^9(3x^4+3-40x^4)}{(x^4+1)^{20}} \\
&= \frac{x^2(x^4+1)^9(3-37x^4)}{(x^4+1)^{20}}
\end{aligned}
$$

Every term in this derivative is positive on $[1,\infty)$ except for $3-37x^4$ which is negative. Thus, $f'(x) < 0$ on $[1,\infty)$ and so $f(x)$ is decreasing on this interval. So we can apply the Integral Test. Using the result from the previous problem, the Integral Test tells us that this series converges.

Part II

10. (10 points) The power series for $f(x) = \dfrac{1}{1+x^2}$ is given by $\dfrac{1}{1+x^2} = \sum_{n=1}^{\infty}(-1)^n x^{2n}$. Use this to get the power series of $\arctan(x)$.

Answer:

Since

$$\arctan(x) = \int \frac{dx}{1+x^2},$$

we can get a Taylor series for $\arctan(x)$ by integrating the one for $1/(1+x^2)$. We have

$$
\begin{aligned}
\frac{1}{1+x^2} &= \sum_{n=0}^{\infty}(-1)^n x^{2n} \\
&= 1 - x^2 + x^4 - x^6 + \cdots
\end{aligned}
$$

so

$$
\begin{aligned}
\arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}
$$

11. (10 points) Consider the series

$$\sum_{n=0}^{\infty} \frac{(-2x)^{3n}}{n!} = 1 - 2^3x^3 + \frac{2^6x^6}{2!} - \frac{2^9x^9}{3!} + \frac{2^{12}x^{12}}{4!} + \cdots$$
$$= 1 - 8x^3 + 32x^6 - \frac{256x^9}{3} + \frac{512x^{12}}{3} + \cdots$$

For which values of x does it converge?

Answer:

We will use the ratio test. The nth term is

$$a_n = \frac{(-2x)^{3n}}{n!}$$

so

$$\frac{a_{n+1}}{a_n} = \frac{(-2x)^{3n+3}}{(n+1)!} \frac{n!}{(-2x)^{3n}}$$
$$= -\frac{8x^3}{n+1}$$

Hence the relevant limit is

$$L = -\lim_{n\to\infty} \frac{8x^3}{n+1} = 0,$$

so the series converges for all x

12. (10 points) Suppose that for the power series $\sum_{n=0}^{\infty} c_n x^n$ centered at $a = 0$, we know $\sum_{n=0}^{\infty} c_n 2^n$ converges and $\sum_{n=0}^{\infty} c_n (-4)^n$ diverges. Then for each of the following series state if it converges, diverges or it is unknown. Justify your answers.

(a) $\sum_{n=0}^{\infty} c_n$

(b) $\sum_{n=0}^{\infty} c_n (-2)^n$

(c) $\sum_{n=0}^{\infty} c_n 5^n$

Answer:

Since $\sum_{n=0}^{\infty} c_n 2^n$ converges, the radius of convergence must be at least 2, and since $\sum_{n=0}^{\infty} c_n(-4)^n$ diverges, the radius of convergence is less than or equal to 4. That is, $2 \leq R \leq 4$. So we can conclude the following:

(a) Since $1 < R$, $\sum_{n=0}^{\infty} c_n$ converges.

(b) It could be that the radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ is 2, and the interval of convergence is $(-2, 2]$, in which case $\sum_{n=0}^{\infty} c_n(-2)^n$ diverges. Otherwise, this series converges. Therefore, from the information we have, we don't know if the series $\sum_{n=0}^{\infty} c_n(-2)^n$ converges or diverges.

(c) Since $5 > R$, $\sum_{n=0}^{\infty} c_n 4^n$ diverges.

13. (10 points)

(a) Find $T_3(x)$, the third degree Taylor polynomial for $f(x) = \frac{1}{x}$ at $a = 1$.

(b) Use Taylor's inequality to estimate the error when $T_3(x)$ is used as an approximation for $f(x)$ on the interval $\frac{1}{2} \leq x \leq \frac{3}{2}$.

Answer:

(a) We should first compute four derivatives and their values at $a = 1$:

$$\begin{array}{rclrcl}
f(x) & = & \frac{1}{x} & f(1) & = & 1 \\
f'(x) & = & \frac{-1}{x^2} & f'(1) & = & -1 \\
f''(x) & = & \frac{2}{x^3} & f''(1) & = & 2 \\
f'''(x) & = & \frac{-6}{x^4} & f'''(1) & = & -6 \\
f^{(4)}(x) & = & \frac{24}{x^5} & f^{(4)}(1) & = & 24
\end{array}$$

Thus, $T_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$.

(b) Notice that $\frac{1}{2} \le x \le \frac{3}{2}$ means that $|x-1| \le \frac{1}{2}$. Also, since $|f^{(4)}(x)| = \frac{24}{x^5}$, we know that $|f^{(4)}(x)| \le \frac{24}{(1/2)^5} = 768$ on the interval $\frac{1}{2} \le x \le \frac{3}{2}$. Thus, by Taylor's Inequality, we have:

$$\begin{aligned} |R_3(x)| &\le \tfrac{768}{4!}|x-1|^4 \\ &\le 32(1/2)^4 \\ &= 2 \end{aligned}$$

14. (10 points) Consider the curve defined by the parametric equations

$$x = t^2 \qquad \text{and} \qquad y = 3t - t^3.$$

(a) For which values of t is the tangent line vertical? Find the corresponding points.

(b) For which values of t is the tangent line horizontal? Find the corresponding points.

Answer:

We have

$$\frac{dx}{dt} = 2t \qquad \text{and} \qquad \frac{dy}{dt} = 3 - 3t^2$$

The tangent line is vertical when $dx/dt = 0$ and $dy/dt \neq 0$, i.e. when $t = 0$ and $(x, y) = (0, 0)$. It is horizotal when $dy/dt = 0$ and $dx/dt \neq 0$, i.e. when $t = \pm 1$ and $(x, y) = \pm(1, 2)$.

15. (10 points)

Set up (but do not evaluate) the integral to find the length of the curve $x = \frac{1}{3}\sqrt{y}(y-3)$ for $4 \le y \le 9$.

Answer:

Since $x = f(y) = \frac{1}{3}\sqrt{y}(y-3) = \frac{1}{3}y^{3/2} - y^{1/2}$, its derivative is $x' = f'(y) = \frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$, and

$$1+(dx/dy)^2 = 1+\left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2 = 1+\frac{y}{4} - \frac{1}{2} + \frac{y^{-1}}{4} = \frac{y}{4} + \frac{1}{2} + \frac{y^{-1}}{4} = \left(\frac{y^{1/2}}{2} + \frac{y^{-1/2}}{2}\right)^2.$$

Thus, the length of the curve is

$$\begin{aligned}
L &= \int_4^9 \sqrt{\left(\frac{y^{1/2}}{2} + \frac{y^{-1/2}}{2}\right)^2}\,dx \\
&= \int_4^9 \left(\frac{y^{1/2}}{2} + \frac{y^{-1/2}}{2}\right) dx \\
&= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_4^9 \\
&= \frac{1}{2}\left(\frac{2}{3}\cdot 9^{3/2} + 2\cdot 9^{1/2} - \frac{2}{3}\cdot 4^{3/2} - 2\cdot 4^{1/2}\right) \\
&= \frac{1}{2}\left(\frac{2}{3}\cdot 27 + 2\cdot 3 - \frac{2}{3}\cdot 8 - 2\cdot 2\right) \\
&= \frac{1}{2}\left(20 - \frac{16}{3}\right) \\
&= \frac{22}{3} \approx 7.333333
\end{aligned}$$

16. (10 points) Find the area of the surface obtained rotating the curve $y = \sqrt{x}$, $0.75 \le x \le 3.75$, about the x-axis.

Answer:

Let $f(x) = \sqrt{x}$, so $1 + (f'(x))^2 = 1 + (\frac{1}{2\sqrt{x}})^2 = 1 + \frac{1}{4x}$. Then the surface area is

$$\begin{aligned}
A &= \int_{0.75}^{3.75} 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx \\
&= 2\pi \int_{0.75}^{3.75} \sqrt{x + \frac{1}{4}}\,dx \\
&= 2\pi \left[\frac{2}{3}\left(x + \frac{1}{4}\right)^{3/2}\right]_{0.75}^{3.75} \\
&= 2\pi \cdot \frac{2}{3} \cdot ((3.75 + 0.25)^{3/2} - (0.75 + 0.25)^{3/2}) \\
&= 2\pi \cdot \frac{2}{3} \cdot (8 - 1) = \frac{28\pi}{3} \approx 29.32153\ldots
\end{aligned}$$

17. (10 points) Consider the polar curve

$$r = 1 + \theta^2, \qquad 0 \le \theta \le 2\pi$$

Find the area of the region bounded by the curve and the ray $\theta = 0$, $r \geq 0$.

Answer:

Using the area formula for polar curves, we get

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} \left(1+\theta^2\right)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1+2\theta^2+\theta^4) \, d\theta \\ &= \frac{1}{2} \cdot \left(\theta + \frac{2\theta^3}{3} + \frac{\theta^5}{5}\right)\Big|_0^{2\pi} \\ &= \pi + \frac{8\pi^3}{3} + \frac{16\pi^5}{5} \end{aligned}$$

18. (10 points) Find the arclength of the curve defined in polar coordinates by the equation

$$r = a\sin\theta$$

where a is a positive constant and $0 \leq \theta \leq \pi$.

Answer:

The curve in question is a circle of radius $a/2$ that goes though the origin and is centered at the point $(0, a/2)$ in rectangular coordinates. Thus we expect the answer to be πa.

We have

$$\begin{aligned} \frac{dr}{d\theta} &= a\cos\theta \\ r^2 + \left(\frac{dr}{d\theta}\right)^2 &= a^2\sin^2\theta + a^2\cos^2\theta \\ &= a^2 \end{aligned}$$

so the arclength is

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^{\pi} a d\theta \\ &= a\pi. \end{aligned}$$