Math 162: Calculus IIA

Final Exam

December 13, 2024

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the box:

Nathanael Grand	MW 9:00 - 10:15 AM	
Doug Ravenel	MW 10:25 - 11:40 AM	
Peter Oberly	MW 12:30 - 1:45 PM	
Peter Oberly	MW 3:25 - 4:40 PM	

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The presence of calculators, cell phones, smart watches, or other electronic devices at this exam is strictly forbidden. If you have your phone with you, you must turn it into a proctor. Failure to do so will be treated as an academic honesty violation.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the space provided at the bottom of each page or half page.**
- **You are responsible for checking that this exam has all 23 pages.**

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates $y = f(x)$ with $0 \le a \le x \le b$:

- about the x-axis: $S = \int_a^b 2\pi|f(x)|\sqrt{1+[f'(x)]^2}\,dx$.
- about the y-axis: $S = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$.

Polar coordinate formulas.

$$x = r\cos(\theta) \qquad r^2 = x^2 + y^2$$

$$y = r\sin(\theta) \qquad \tan(\theta) = y/x$$

Note: $\theta = \arctan(y/x)$ when $x > 0$, and $\theta = \arctan(y/x) + \pi$ when $x < 0$.

Changing θ by any multiple of 2π does not change the location of the point.
Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar cooridinates for $r = f(\theta)$, with $\alpha \leq \theta \leq \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta.$$

Arc length formulas:

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

- Polar coordinates, $r = f(\theta)$, $\alpha \leq \theta \leq \beta$:

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

- Parametric equations, $x = x(t), y = y(t)$ with $a \leq t \leq b$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_n(x)| \leq \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (20 points) Fix $b > 0$. Compute the arc length of the polar curve $r = e^{b\theta}$, where $0 \leq \theta \leq \pi$.

ANSWER:

2. (20 points)

Consider the function $y = \sqrt{x+1}$ on the interval $[1, 5]$.

(a) **(10 Points)** Compute the volume of the region bound by the curves $y = \sqrt{x+1}$, $x = 1, x = 5$ and the x-axis, revolved about the x-axis.

ANSWER:

(b) **(10 Points)** Compute the surface area of the region bound by the curves $y = \sqrt{x+1}$, $x = 1$, and $x = 5$, revolved about the x-axis.

ANSWER:

3. (20 points) The ionization energy of an atom is the energy required to free an electron from the atom. For the hydrogen atom, this is given approximately by

$$\int_R^\infty \frac{1}{2}\frac{kq^2}{r^2}dr$$

Where R is the initial radius of the electron, k is a constant, and q is the magnitude of the charges of the electron and proton. Compute the above integral. Your answer should be in terms of R, k and q. Don't worry about units.

ANSWER:

4. (20 points) Evaluate the following integrals.

(a) **(10 points.)** $\int x^2 \ln(x)\, dx$

ANSWER:

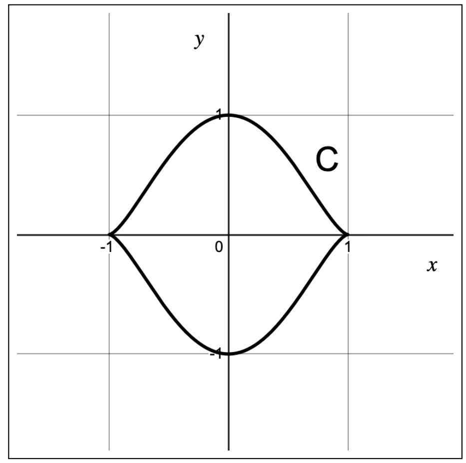
(b) **(10 points)** $\int \frac{2x+1}{x^3+x}\,dx$

ANSWER:

5. (20 points) A curve C is defined by the parametric equation

$$x = x(t) = -\cos(t), \qquad y = y(t) = \sin^3(t), \qquad 0 \le t \le 2\pi.$$

Find the area of the region enclosed by this curve.



ANSWER:

Part B

6. (10 points) Find the sum of the series $\sum_{n=3}^{\infty} \frac{2^n - 7^{1-n}}{e^n}$.

ANSWER:

7. (30 points) (10 points each.) For each of the following series, determine if the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer with an appropriate convergence/divergence test.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+1}$

ANSWER:

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

ANSWER:

(c) $\sum_{n=1}^{\infty} \big(\cos(1/n) - 1 \big)^n$

ANSWER:

8. (20 points) Let

$$f(x) = \frac{x^2}{1 + 2x}.$$

(a) **(10 points.)** Find the Taylor series of $f(x)$ centered at $x = 0$.

ANSWER:

(b) **(10 points.)** Find the radius of convergence.

ANSWER:

9. (20 points) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^{n+2}}$$

ANSWER:

10. (20 points) For $|x| < 1$, set $F(x) = \displaystyle\int_0^x \frac{\ln(1+t)}{t}\,dt$.

(a) **(10 points.)** Represent $F(x)$ as a power series. [You do **not** need to find the radius or interval of convergence.]

ANSWER:

(b) **(10 points.)** Use your answer from part (a) to estimate the value of $F(1/2) = \int_0^{1/2} \frac{\ln(1+t)}{t}\, dt$ with an accuracy of 1/100. [Hint: use the error bound for alternating series.]

ANSWER:

Scratch paper

Scratch paper

Scratch paper