Math 162: Calculus IIA

Final Exam ANSWERS

December 3, 2024

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2 + 1}{2u} \qquad \tan(\theta) = \frac{u^2 - 1}{2u}$$

Area of surface of revolution in rectangular coordinates $y = f(x)$ with $0 \le a \le x \le b$:

- about the x-axis: $S = \int_a^b 2\pi|f(x)|\sqrt{1 + [f'(x)]^2}\,dx$.
- about the y-axis: $S = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}\,dx$.

Polar coordinate formulas.

$$x = r\cos(\theta) \qquad r^2 = x^2 + y^2$$

$$y = r\sin(\theta) \qquad \tan(\theta) = y/x$$

Note: $\theta = \arctan(y/x)$ when $x > 0$, and $\theta = \arctan(y/x) + \pi$ when $x < 0$.

Changing θ by any multiple of 2π does not change the location of the point.
Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$, with $\alpha \le \theta \le \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta.$$

Arc length formulas:

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

- Polar coordinates, $r = f(\theta)$, $\alpha \le \theta \le \beta$:

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

- Parametric equations, $x = x(t), y = y(t)$ with $a \le t \le b$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_n(x)| \leq \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (20 points) Fix $b > 0$. Compute the arc length of the polar curve $r = e^{b\theta}$, where $0 \leq \theta \leq \pi$.

Answer:

Observe that $r' = be^{b\theta}$ and so

$$\begin{aligned} L &= \int_0^\pi \sqrt{(r)^2 + (r')^2}\, d\theta \\ &= \int_0^\pi \sqrt{(e^{b\theta})^2 + b^2(e^{b\theta})^2}\, d\theta \\ &= \int_0^\pi \sqrt{(e^{b\theta})^2(1+b^2)}\, d\theta \\ &= \int_0^\pi e^{b\theta}\sqrt{1+b^2}\, d\theta \\ &= \sqrt{1+b^2} \int_0^\pi e^{b\theta}\, d\theta \\ &= \sqrt{1+b^2} \left[\frac{e^{b\theta}}{b}\right]_{\theta=0}^{\pi} \\ &= \sqrt{1+b^2}\, \frac{e^{b\pi}-1}{b}. \end{aligned}$$

2. (20 points)

Consider the function $y = \sqrt{x+1}$ on the interval $[1, 5]$.

(a) **(10 Points)** Compute the volume of the region bound by the curves $y = \sqrt{x+1}$, $x = 1, x = 5$ and the x-axis, revolved about the x-axis.

Answer:

We can use the Disk (aka Washer) Method:

$$\begin{aligned}\int_1^5 \pi(\sqrt{x+1})^2 dx &= \int_1^5 \pi(x+1)dx \\ &= \pi\left[\frac{x^2}{2} + x\right]_1^5 \\ &= 16\pi\end{aligned}$$

(b) **(10 Points)** Compute the surface area of the region bound by the curves $y = \sqrt{x+1}$, $x = 1$, and $x = 5$, revolved about the x-axis.

Answer:

The derivative of $\sqrt{x+1}$ is $1/2\sqrt{x+1}$. So the formula for the surface area is:

$$\begin{aligned}\int_1^5 2\pi\sqrt{x+1} \cdot \sqrt{1 + \frac{1}{4(x+1)}} dx &= 2\pi \int_1^5 \sqrt{x + 1 + \frac{1}{4}} dx \\ &= 2\pi \left[\frac{2}{3}(x + \frac{5}{4})^{3/2}\right]_1^5 \\ &= \frac{4}{3}\pi\left(\frac{125}{8} - \frac{27}{8}\right) \\ &= \frac{49\pi}{3}\end{aligned}$$

3. (20 points) The ionization energy of an atom is the energy required to free an electron from the atom. For the hydrogen atom, this is given approximately by

$$\int_R^\infty \frac{1}{2}\frac{kq^2}{r^2} dr$$

Where R is the initial radius of the electron, k is a constant, and q is the magnitude of the charges of the electron and proton. Compute the above integral. Your answer should be in terms of R, k and q. Don't worry about units.

Answer:

$$\int_R^\infty \frac{1}{2}\frac{kq^2}{r^2}dr = \lim_{t\to\infty}\int_R^t \frac{1}{2}\frac{kq^2}{r^2}dr = \lim_{t\to\infty}\frac{kq^2}{2}\int_R^t \frac{1}{r^2}dr$$

This then simplifies to

$$\frac{kq^2}{2}\left(\lim_{t\to\infty} -\frac{1}{r}\bigg|_R^t\right) = \frac{kq^2}{2}\left(\lim_{t\to\infty}\frac{1}{R}-\frac{1}{t}\right) = \frac{kq^2}{2R}$$

4. (20 points) Evaluate the following integrals.

(a) **(10 points.)** $\int x^2 \ln(x)\, dx$

Answer:

Integrate by parts: Let $u = \ln(x), du = dx/x$ and $dv = x^2\ dx, v = \frac{1}{3}x^3$. We get

$$\begin{aligned}
\int x^2 \ln(x)\ dx &= \int u\ dv \\
&= uv - \int v\ du \\
&= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^3 \frac{dx}{x} \\
&= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^2\ dx \\
&= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C.
\end{aligned}$$

(b) **(10 points)** $\displaystyle\int \frac{2x+1}{x^3+x}\,dx$

Answer:

Factor the denominator as $x(x^2+1)$. The partial fraction decomposition takes the form

$$\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

So

$$2x+1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

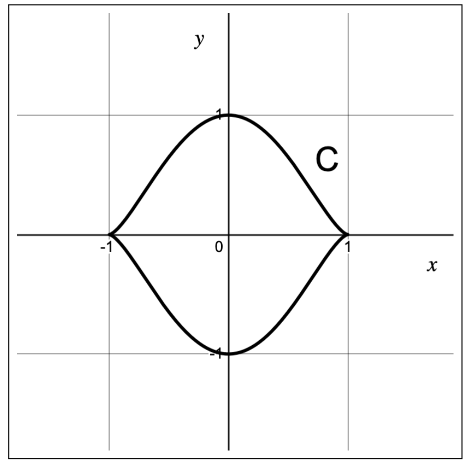
Equating coefficients of like powers of x shows that $A+B=0, C=2$ and $A=1$ (so $B=-1$.) Therefore

$$\begin{aligned}
\int \frac{2x+1}{x(x^2+1)} &= \int \frac{1}{x} + \frac{-x+2}{x^2+1}\,dx \\
&= \int \frac{1}{x} - \frac{x}{x^2+1} + \frac{2}{x^2+1}\,dx \\
&= \ln|x| - \frac{1}{2}\ln|1+x^2| + 2\arctan(x) + C.
\end{aligned}$$

5. (20 points) A curve C is defined by the parametric equation

$$x = x(t) = -\cos(t), \qquad y = y(t) = \sin^3(t), \qquad 0 \le t \le 2\pi.$$

Find the area of the region enclosed by this curve.



Answer:

We find the area A under the curve and above the x-axis from $x = -1$ to $x = 0$, and then multiply by 4 to find the total area enclosed by this curve. [Alternatively, we could find the area under the curve from $x = -1$ to $x = 1$ and multiply by 2.]

We have $x = -1$ when $t = 0$ and $x = 0$ when $t = \pi/2$. Also, $dx = \sin(t)dt$ and so

$$\begin{aligned}
A &= \int_{-1}^{0} y\, dx \\
&= \int_0^{\pi/2} \sin^3(t)(\sin(t))\, dt \\
&= \int_0^{\pi/2} \sin^4(t)\, dt \\
&= \int_0^{\pi/2} \left(\frac{1-\cos(2t)}{2}\right)^2\, dt \\
&= \int_0^{\pi/2} \frac{1}{4}\left(1 - 2\cos(2t) + \cos^2(2t)\right)\, dt \\
&= \frac{1}{4}\int_0^{\pi/2} 1 - 2\cos(2t) + \frac{1+\cos(4t)}{2}\, dt \\
&= \frac{1}{4}\int_0^{\pi/2} \frac{3}{2} - 2\cos(2t) + \cos(4t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[\frac{3}{2}t - 2\frac{\sin(2t)}{2} + \frac{\sin(4t)}{4}\right]_{t=0}^{\pi/2} \\
&= \frac{1}{4}\cdot\frac{3\pi}{4} \\
&= \frac{3\pi}{16}.
\end{aligned}
$$

Thus the area enclosed by the region is $4A = 3\pi/4$.

Part B

6. (10 points) Find the sum of the series $\displaystyle\sum_{n=3}^{\infty} \frac{2^n - 7^{1-n}}{e^n}$.

Answer:

Applying the geometric series formula gives

$$\sum_{n=3}^{\infty} 2^n/e^n = \sum_{n=3}^{\infty} (2/e)^n = \frac{(2/e)^3}{1-(2/e)},$$

and

$$\sum_{n=3}^{\infty} 7^{1-n}/e^n = \sum_{n=3}^{\infty} 7/(7e)^n = 7\frac{(7e)^{-3}}{1-(7e)^{-1}}$$

so that

$$\sum_{n=3}^{\infty} \frac{2^n - 7^{1-n}}{e^n} = \sum_{n=3}^{\infty} \frac{2^n}{e^n} - \sum_{n=3}^{\infty} \frac{7^{1-n}}{e^n} = \frac{(2/e)^3}{1-(2/e)} - 7\frac{(7e)^{-3}}{1-(7e)^{-1}}.$$

7. (30 points) (10 points each.) For each of the following series, determine if the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer with an appropriate convergence/divergence test.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3+1}$

Answer:

Conditionally convergent.
Observe that $n^2/(n^3+1) \to 0$ as $n \to \infty$, and that if $f(x) = (x^2)/(x^3+1)$ then

$$\begin{aligned} f'(x) &= \frac{(2x)(x^3+1) - (x^2)(3x^2)}{(x^3+1)^2} \\ &= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} \\ &= \frac{-x^4 + 2x}{(x^3+1)^2}. \end{aligned}$$

For x sufficiently large, we will have $x^4 > 2x$. ($x \geq 2$ suffices.) Therefore $f'(x) < 0$ for all large x and so $f(n+1) < f(n)$ for all sufficiently large n. So the series satisfies the conditions of the alternating series test and is therefore convergent.

To see that $\sum(n^2)/(n^3+1)$ does not converge - and so the series is not absolutely convergent - apply the limit comparison test with the harmonic series $\sum 1/n$:

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} n^3/(n^3+1) = 1 > 0$$

so that $\sum n^2/(n^3+1)$ and $\sum 1/n$ either both converge or both diverge. Since $\sum 1/n$ is divergent, then so is $\sum n^2/(n^3+1)$. Alternatively, one can also use the integral test here.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

Answer:

Absolutely convergent.
Use the integral test:

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx &= \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du \quad \text{(via the substitution } u = \ln x.\text{)}\\ &= \lim_{t\to\infty} \frac{-1}{u}\Big]_{\ln(2)}^{t}\\ &= \lim_{t\to\infty} \frac{-1}{t} + \frac{1}{\ln(2)}\\ &= 1/\ln(2).\end{aligned}$$

So the integral test implies $\sum 1/(n(\ln n)^2$ converges. Since all of the terms in this series are positive then the series converges absolutely.

(c) $\sum_{n=1}^{\infty} \big(\cos(1/n) - 1\big)^n$

Answer:

Absolutely convergent. Apply the root test:

$$\lim_{n\to\infty} |(\cos(1/n) - 1)^n|^{1/n} = \lim_{n\to\infty} |\cos(1/n) - 1| = \cos(0) - 1 = 0.$$

Since this limit is strictly less than 1, the series converges absolutely.

8. (20 points) Let

$$f(x) = \frac{x^2}{1+2x}.$$

(a) **(10 points.)** Find the Taylor series of $f(x)$ centered at $x = 0$.

Answer:

Write

$$\begin{aligned} \frac{x^2}{1+2x} &= \frac{x^2}{1-(-2x)} \\ &= x^2 \sum_{n=0}^{\infty} (-2x)^n \qquad \text{(using geometric series expansion)} \\ &= \sum_{n=0}^{\infty} (-2)^n x^{n+2}. \end{aligned}$$

(b) **(10 points.)** Find the radius of convergence.

Answer:

One way is to note that $f(x)$ is not defined at $x = -1/2$. This is a distance of $1/2$ away from the center $x = 0$. Thus, the radius of convergence will be $1/2$. Alternatively, you can use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+3}}{(-2)^n x^{n+2}} \right| = |2x|$$

For the series to converge by the Ratio test, we must have $|2x| < 1$, which means $|x| < 1/2$. Thus, again, the radius of convergence is $1/2$.

9. (20 points) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^{n+2}}$$

Answer:

The ratio test will be applied

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2 3^{n+3}}}{\frac{x^n}{n^2 3^{n+2}}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2 3^{n+3}} \frac{n^2 3^{n+2}}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{3} \frac{n^2}{(n+1)^2} \right|$$

This simplifies to

$$\frac{|x|}{3} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{|x|}{3}$$

The ratio test provides absolute convergence as long as this limit is strictly less than 1, and divergence if the limit is strictly larger than 1. For convergence:

$$\frac{|x|}{3} < 1 \iff |x| < 3$$

Therefore the radius of convergence is 3. To get the interval, the edge cases $|x| = 3$ must be checked. This corresponds to $x = \pm 3$. When $x = -3$ we check

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^{n+2}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^2} = \frac{1}{9} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

The series above is alternating, with $\frac{1}{n^2}$ decreasing, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$. AST implies convergence. Then, we check $x = 3$:

$$\sum_{n=1}^{\infty} \frac{(3)^n}{n^2 3^{n+2}} = \frac{1}{9} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This converges by a p-test, ($p = 2$, $2 > 1$). This gives us the interval of convergence: $I = [-3, 3]$.

10. (20 points) For $|x| < 1$, set $F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt$.

(a) **(10 points.)** Represent $F(x)$ as a power series. [You do **not** need to find the radius or interval of convergence.]

Answer:

Using the Macluarin series for $\ln(1+x)$, we get

$$\begin{aligned}\frac{\ln(1+t)}{t} &= \frac{1}{t}\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{n+1}}{n+1}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n+1}\end{aligned}$$

and so

$$\begin{aligned}F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^x t^n\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)(n+1)}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{(n+1)^2}\\ &= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\end{aligned}$$

(b) **(10 points.)** Use your answer from part (a) to estimate the value of $F(1/2) = \int_0^{1/2} \frac{\ln(1+t)}{t}\, dt$ with an accuracy of 1/100. [Hint: use the error bound for alternating series.]

Answer:

From part (a), we can write $F(1/2)$ as an alternating series:

$$F(1/2) = \int_0^{1/2} \frac{\ln(1+t)}{t}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{n+1}}{(n+1)^2}$$
$$= 1/2 - \frac{(1/2)^2}{2^2} + \frac{(1/2)^3}{3^2} - \frac{(1/2)^4}{4^2} + \ldots$$
$$= \frac{1}{2} - \frac{1}{4\cdot 4} + \frac{1}{8\cdot 9} - \frac{1}{16\cdot 16} + \ldots$$

The alternating series estimation theorem then implies

$$\left| F(1/2) - \left(\frac{1}{2} - \frac{1}{16} + \frac{1}{72}\right) \right| \leq \frac{1}{16^2} < \frac{1}{100}$$

So that $F(1/2) \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{72}$ with an error of at most $1/16^2 < 1/100$.

Scratch paper

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