Math 162: Calculus IIA

Final Exam

December 17, 2023

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the box:

Firdavs Rakhmonov	MW 9:00 - 10:15 AM	
Doug Ravenel	MW 10:25 - 11:40 AM	
Peter Oberly	MW 12:30 - 1:45 PM	
Sefika Kuzgun	MW 3:25 - 4:40 PM	

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The presence of calculators, cell phones and other electronic devices at this exam is strictly forbidden and WILL BE TREATED AS AN ACADEMIC HONESTY VIOLATION.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Put your answers in the space provided at the bottom of each page or half page. SIMPLIFY YOUR ANSWERS AS MUCH AS POSSIBLE.**
- **Part A (problems 1–6) covers the same material as the two midterms, and Part B (problems 7–10) covers additional material. Letter grades will be computed for the two parts separately. Part B will count for 23% of your course grade. Part A will count for at least 8% of your course grade. If your letter grade on part A is better than your lowest midterm letter exam grade, then it will replace that midterm exam grade and count for 22% of your course grade.**
- **You are responsible for checking that this exam has all 24 pages.**

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$

$$\pi + \arctan(y/x) \text{for } x < 0$$

$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$

$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$

$$\text{undefined} \text{for } (x, y) = (0, 0)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \le t \le b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \le M$ for suitable x, then

$$|R_n(x)| \le \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (10 points)

Evaluate the integral

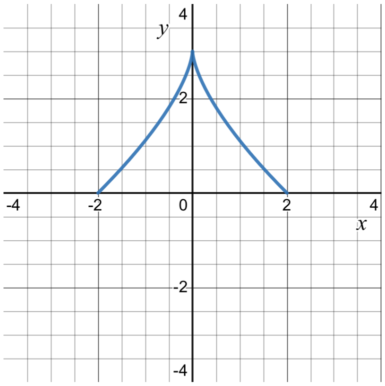
$$\int \arctan(2x)dx.$$

ANSWER:

2. (20 points) Find the arc length of the following parametric curve.

$$x = 2\cos^3(t), \quad y = 3\sin^2(t), \quad 0 \le t \le \pi.$$

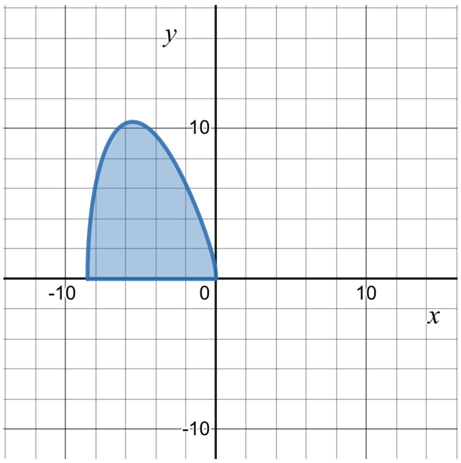
Hint: Find the arc-length for $0 \le t \le \pi/2$ and then multiply your result by 2.



ANSWER:

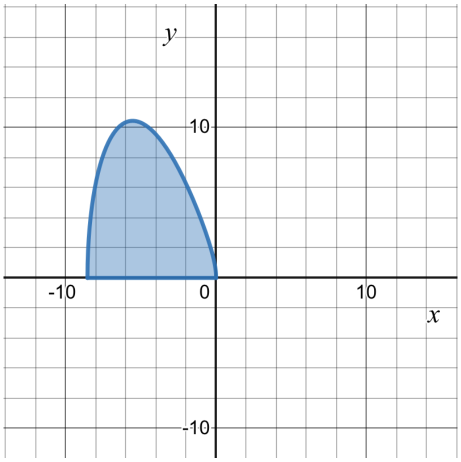
3. (20 points)

(a) (10 points) Fix a positive number t. Compute the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x(x-1)(x+t)}$, $y = 0$, about the x-axis. Your answer should be a function of t.



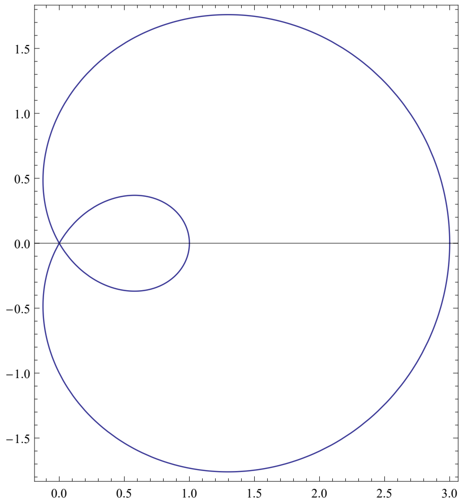
ANSWER:

(b) (10 points) Fix a positive number t. Set up the integral for the volume of the region bounded by $y = \sqrt{x(x-1)(x+t)}$, $y = 0$ and rotated around the line $x = 1$. Your integral should depend on t. Do not evaluate the integral.



ANSWER:

4. (20 points) Find the area inside the outer (larger) loop but outside the inner (smaller) loop of the limaçon $r = 1 + 2\cos(\theta)$.



ANSWER:

5. (15 points) Compute the following indefinite integral:

$$\int \frac{x^2 + 3x}{x^2 - 1}\, dx$$

ANSWER:

6. (15 points) Compute the following indefinite integral:

$$\int \frac{x^2\,dx}{(1-x^2)^{3/2}}$$

ANSWER:

Part B

7. (20 points)

(a) (10 points) Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5}$$

is absolutely convergent, conditionally convergent, or divergent.

ANSWER:

(b) (10 points) Estimate the sum of the series with an accuracy of $.01 = 1/100$.

ANSWER:

8. (20 points)

(a) (10 points) Find a power series representation centered at 1 as well as the radius and interval of convergence for the function

$$f(x) = \frac{x-1}{x+2}.$$

ANSWER:

(b) (10 points) Write the following integral as a power series in $x-1$. What is the radius of convergence of this power series?

$$\int \frac{x-1}{x+2} dx$$

ANSWER:

9. (20 points)

(a) (10 points) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

ANSWER:

(b) (10 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

ANSWER:

10. (20 points)

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent. Give reasoning for your answers.

(a) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$

ANSWER:

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2}$$

ANSWER:

This is scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Second scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Third scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Fourth scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Fifth scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.