Math 162: Calculus IIA

Final Exam

December 18, 2022

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The presence of calculators, cell phones and other electronic devices at this exam is strictly forbidden and WILL BE TREATED AS AN ACADEMIC HONESTY VIOLATION.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given. Put your answers in the space provided at the bottom of each page or half page. SIMPLIFY YOUR ANSWERS AS MUCH AS POSSIBLE.**
- **Part A (problems 1–6) covers the same material as the two midterms, and Part B (problems 7–10) covers additional material. Letter grades will be computed for the two parts separately. Part B will count for 20% of your course grade. Part A will count for at least 10% of your course grade. If your letter grade on part A is better than your lowest midterm letter exam grade, then it will replace that midterm exam grade and count for 30% of your course grade.**
- **You are responsible for checking that this exam has all 17 pages.**

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

MORE FORMULAS FOR YOUR ENJOYMENT

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$

$$\pi + \arctan(y/x) \text{for } x < 0$$

$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$

$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$

$$\text{undefined} \text{for } (x, y) = (0, 0)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \le t \le b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_n(x)| \leq \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (20 points)

(a) (10 points) The form of the partial fraction decomposition of the function is given below:

$$\frac{6x^2 + x - 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Find the coefficients A, B and C.

ANSWER:

(b) (10 points) Evaluate the following integral:

$$\int \frac{6x^2 + x - 1}{x^3 + x} dx.$$

ANSWER:

2. (10 points) Compute the following integral:

$$\int xe^{4x+2}dx$$

ANSWER:

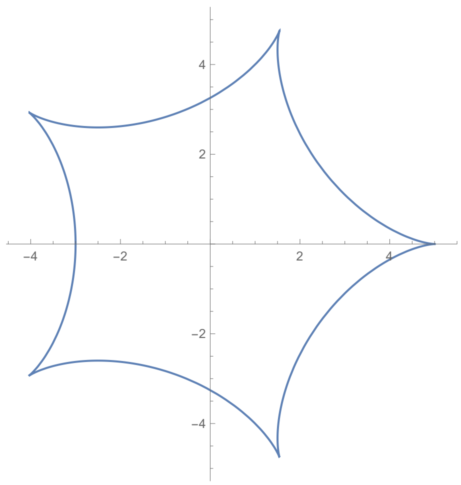
3. (20 points) Find the arc-length of the parametric curve

$$x = 4\cos t + \cos 4t\,,\ \ y = 4\sin t - \sin 4t\,,\ \ 0 \le t \le 2\pi$$

by doing it for $0 \le t \le 2\pi/5$ and multiplying your answer by 5.

YOU MAY WANT TO USE THE TRIG IDENTITIES $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ AND $\sin^2\theta = (1-\cos 2\theta)/2$.

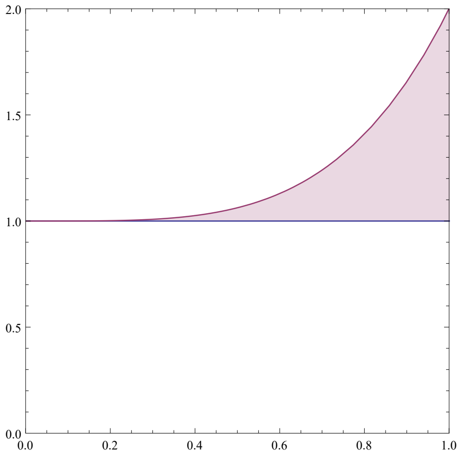
The curve for $0 \le t \le 2\pi$ is pictured below.



ANSWER:

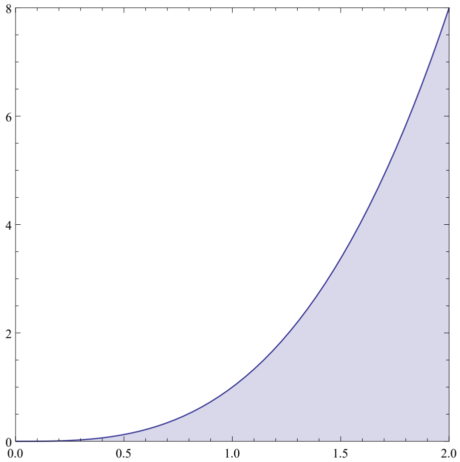
4. (20 points)

(a) (10 points) Compute the volume of a region bounded by the curves $y = x^4 + 1$, $y = 1$ and $x = 1$ and rotated around the y-axis.



ANSWER:

(b) (10 points) Set up the integral for the volume of the region bounded by $y = x^3$, $y = 0$ and $x = 2$ and rotated around line $x = 2$. Use the shell method. Do not evaluate the integral.



ANSWER:

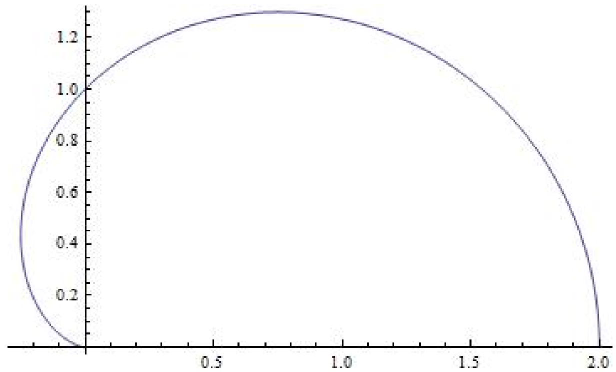
5. (15 points) Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx.$$

ANSWER:

6. (15 points)

The cardioid is the curve defined in polar coordinates by $r = 1 + \cos\theta$. Find the area of the region bounded above by the cardioid and below by the x-axis.



ANSWER:

Part B

7. (20 points) Let q be a positive (greater than 0) real number.

(a) (10 points)

Find the radius of convergence of the series $\sum_{n=0}^{\infty}(-1)^n\left(\frac{q}{q+1}\right)^n(x-a)^n$.

ANSWER:

(b) (10 points)

Find the interval of convergence of the series $\sum_{n=0}^{\infty}(-1)^n\left(\frac{q}{q+1}\right)^n(x-a)^n$.

ANSWER:

8. (20 points)

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent. Give reasoning for your answers.

(a) (10 points)

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

ANSWER:

$=\infty$

But the series is convergent by the alternating series test since $a_n = \frac{1}{n\ln(n)}$ is positive on $n \geq 3$ and decreasing (the denominator gets larger, so the terms get smaller) and

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0.$$

Therefore the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges conditionally.

(10 points)

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1-4n}{1+3n}\right)^{2n}$$

ANSWER:

9. (20 points)

(a) (10 points) Evaluate the following indefinite integral as an infinite series:

$$\int e^{-x^2}\,dx$$

ANSWER:

(b) (10 points) Use your answer from part (a) to approximate the value of the following definite integral so that the error is less than 1/200:

$$\int_0^1 e^{-x^2}\, dx$$

Express the relevant partial sum of the series as a fraction with whole numerator and denominator. (Hint: use the error bound for alternating series.)

ANSWER:

10. (20 points)

Calculate the Taylor series for the following functions using any technique that you choose. Show your work. You should have a single sum in your answers. You do not need to specify the radius/interval of convergence.

(a) (10 points) $f(x) = \dfrac{1}{1-2x} - \dfrac{1}{1+2x}$ centered at $x = 0$ $(a = 0)$.

ANSWER:

(b) (10 points) $f(x) = e^{-2x}$ centered at $x = 2 \quad (a = 2)$.

ANSWER:

This is scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Second scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Third scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Fourth scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.