Math 162: Calculus IIA

Final Exam ANSWERS

December 19, 2022

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$
$$\pi + \arctan(y/x) \text{for } x < 0$$
$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$
$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$
$$\text{undefined} \text{for } (x, y) = (0, 0)$$
$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \le t \le b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

INFINITE SERIES FORMULAS

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_n(x)| \leq \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (20 points)

(a) (10 points) The form of the partial fraction decomposition of the function is given below:

$$\frac{6x^2+x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Find the coefficients A, B and C.

Answer:

$$\frac{6x^2+x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$
$$6x^2+x-1 = A(x^2+1)+(Bx+C)x = (A+B)x^2+Cx+A.$$

Hence $A+B=6$, $C=1$ and $A=-1$, which implies $\boxed{A=-1,\ B=7,\ C=1}$.

(b) (10 points) Evaluate the following integral:

$$\int \frac{6x^2+x-1}{x^3+x}dx.$$

Answer:

From part (a):

$$\int \frac{6x^2+x-1}{x^3+x}dx = \int \frac{-1}{x} + \frac{7x}{x^2+1} + \frac{1}{x^2+1}dx$$
$$\int \frac{-1}{x}dx = -\ln|x| + C.$$

For $\int \frac{7x}{x^2+1}dx$ we will use substitution $u = x^2+1$:

$$\int \frac{7x}{x^2+1}dx = \frac{7}{2}\int \frac{1}{u}du = \frac{7}{2}\ln(u) + C = \frac{7}{2}\ln|x^2+1| + C$$

and finally

$$\int \frac{1}{x^2+1}dx = \arctan(x) + C.$$

So in total,

$$\int \frac{6x^2+x-1}{x^3+x}dx = -\ln|x| + \frac{7}{2}\ln|x^2+1| + \arctan(x) + C.$$

2. (10 points) Compute the following integral:

$$\int xe^{4x+2}dx$$

Answer:

We will do integration by parts: $u = x, dv = e^{4x+2}dx, \Rightarrow du = dx, v = \frac{1}{4}e^{4x+2}$.

$$\begin{aligned}\int xe^{4x+2}dx &= \frac{x}{4}e^{4x+12} - \int \frac{1}{4}e^{4x+2}dx \\ &= \frac{x}{4}e^{4x+2} - \frac{1}{16}e^{4x+2} + C.\end{aligned}$$

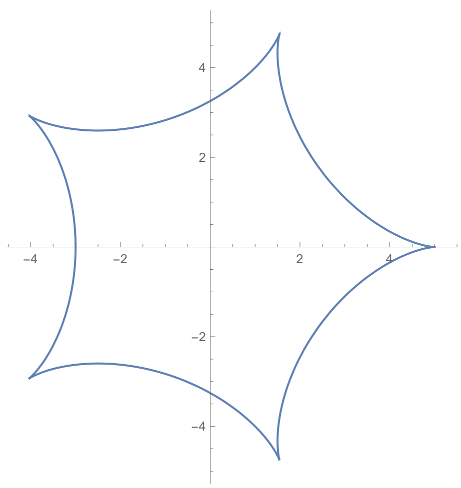
3. (20 points) Find the arc-length of the parametric curve

$$x = 4\cos t + \cos 4t\,,\;\; y = 4\sin t - \sin 4t\,,\;\; 0 \le t \le 2\pi$$

by doing it for $0 \le t \le 2\pi/5$ and multiplying your answer by 5.

YOU MAY WANT TO USE THE TRIG IDENTITIES $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ AND $\sin^2\theta = (1-\cos 2\theta)/2$.

The curve for $0 \le t \le 2\pi$ is pictured below.



Answer:

We have

$$dx/dt = -4(\sin t + \sin 4t) \qquad \text{and} \qquad dy/dt = 4(\cos t - \cos 4t).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= 16(\sin t + \sin 2t)^2 + 16(\cos t - \cos 4t)^2 \\
&= 16(\sin^2 t + 2\sin t \sin 4t + \sin^2 4t + \cos^2 t - 2\cos t \cos 4t + \cos^2 4t) \\
&= 16(2 - 2\cos 5t) = 32(1 - \cos 5t) \\
&\qquad \text{since } \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= 64\left(\frac{1 - \cos 5t}{2}\right) \\
&= 64\sin^2(5t/2),
\end{aligned}$$

so

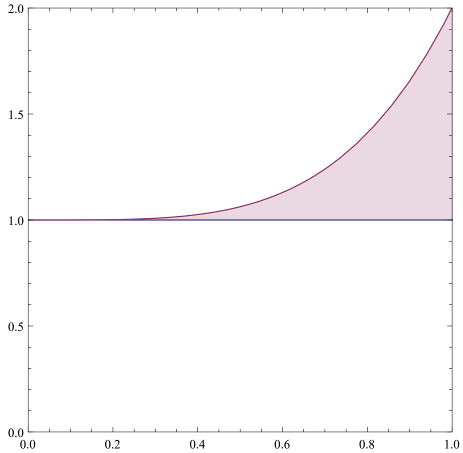
$$\frac{ds}{dt} = 8|\sin(5t/2)|.$$

By the arc length formula, we have

$$\begin{aligned}
L &= 5\int_0^{2\pi/5} ds = 40\int_0^{2\pi/5} \sin(5t/2)\,dt \\
&= 16\int_0^{\pi} \sin u\,du, \qquad \text{where } u = 5t/2, \text{ so } dt = 2du/5 \\
&= -16\cos u\Big|_0^{\pi} = 32.
\end{aligned}$$

4. (20 points)

(a) (10 points) Compute the volume of a region bounded by the curves $y = x^4 + 1$, $y = 1$ and $x = 1$ and rotated around the y-axis.

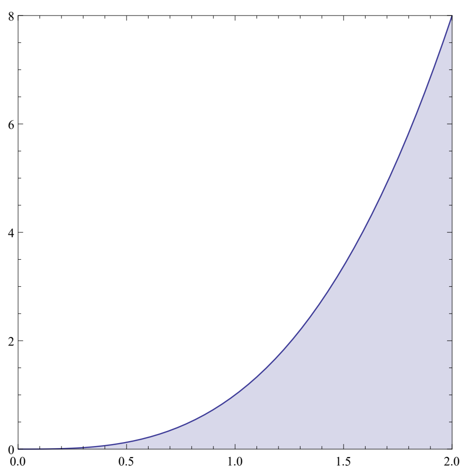


Answer:

Using the shell method we have shells of radius x, thickness dx and height $(x^4+1)-1=x^4$. Therefore

$$V = \int_0^1 2\pi x \cdot x^4 dx = 2\pi \frac{x^6}{6}\bigg|_0^1 = \frac{\pi}{3}$$

(b) (10 points) Set up the integral for the volume of the region bounded by $y = x^3$, $y = 0$ and $x = 2$ and rotated around line $x = 2$. Use the shell method. Do not evaluate the integral.



Answer:

Using the shell method we have shells of radius $(2-x)$, thickness dx and height x^3. Thus the volume is

$$V = \int_0^2 2\pi(2-x)x^3\, dx.$$

5. (15 points) Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx.$$

Answer:

Use the substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta.$$

So

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = \int \frac{1}{16\tan^2\theta\, 4\sec\theta} 4\sec^2\theta\,d\theta$$
$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{16}\left[-\frac{1}{\sin\theta}\right] + C = -\frac{1}{16\sin\theta} + C.$$

From $\tan\theta = \frac{x}{4}$, by drawing a right triangle with one angle θ, we can check that

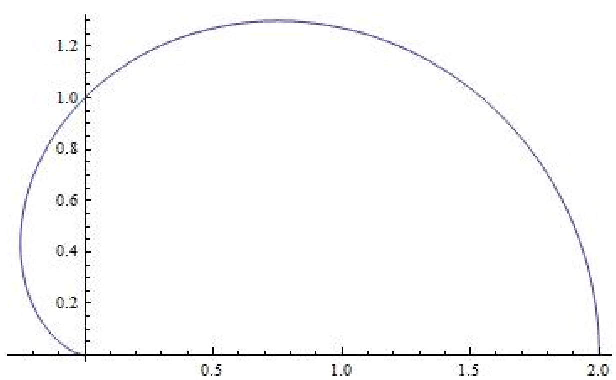
$$\sin\theta = \frac{x}{\sqrt{x^2+16}},$$

so the answer becomes

$$-\frac{1}{16}\frac{\sqrt{x^2+16}}{x} + C.$$

6. (15 points)

The cardioid is the curve defined in polar coordinates by $r = 1+\cos\theta$. Find the area of the region bounded above by the cardioid and below by the x-axis.



Answer:

Solution: It is easily verified that the region R bounded above by the cardioid and below by the x-axis is given by

$$R = \{(r,\theta) : 0 \le r \le 1 + \cos\theta, 0 \le \theta \le \pi\}.$$

We use the formula for area inside a polar curve to compute that the area A of the region R is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^\pi (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta \\ &= \frac{1}{2}\int_0^\pi \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)\,d\theta = \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi \\ &= \frac{3\pi}{4}. \end{aligned}$$

Part B

7. (20 points) Let q be a positive (greater than 0) real number.

(a) (10 points)

Find the radius of convergence of the series $\displaystyle\sum_{n=0}^\infty (-1)^n \left(\frac{q}{q+1}\right)^n (x-a)^n$.

Answer:

Applying the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\left(\frac{q}{q+1}\right)^{n+1}(x-a)^{n+1}}{(-1)^n\left(\frac{q}{q+1}\right)^n (x-a)^n}\right| = \lim_{n\to\infty}\frac{q}{q+1}|x-a| = \frac{q}{q+1}|x-a|$$

As $\frac{q}{q+1}|x-a| < 1$ if and only if $|x-a| < \frac{q+1}{q}$, we can conclude that the radius of convergence is $\frac{q+1}{q}$.

(b) (10 points)

Find the interval of convergence of the series $\displaystyle\sum_{n=0}^\infty (-1)^n \left(\frac{q}{q+1}\right)^n (x-a)^n$.

Answer:

To determine the interval of convergence, we plug in $x = a \pm \frac{q+1}{q}$ into the original expression. For $x = a + \frac{q+1}{q}$, the series becomes

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{q}{q+1}\right)^n \left(\frac{q+1}{q}\right)^n = \sum_{n=0}^{\infty}(-1)^n,$$

which diverges. For $x = a - \frac{q+1}{q}$, the series becomes

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{q}{q+1}\right)^n \left(-\frac{q+1}{q}\right)^n = \sum_{n=0}^{\infty} 1,$$

which also diverges. Hence, the interval of convergence is

$$\left(a - \frac{q+1}{q}, a + \frac{q+1}{q}\right).$$

8. (20 points)

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent. Give reasoning for your answers.

(a) (10 points)

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)}$$

Answer:

The series is not absolutely convergent: $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$ diverges by integral test since

$$\int_3^{\infty} \frac{1}{x\ln(x)}dx = \lim_{t\to\infty}\int_3^t \frac{1}{x\ln(x)}dx = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(3)) = \infty$$

But the series is convergent by the alternating series test since $a_n = \frac{1}{n\ln(n)}$ is positive on $n \geq 3$ and decreasing (the denominator gets larger, so the terms get smaller) and

$$\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0.$$

Therefore the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converges conditionally.

(b) (10 points)

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1-4n}{1+3n} \right)^{2n}$$

Answer:

We apply the root test.

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left| (-1)^n \left(\frac{1-4n}{1+3n} \right)^{2n} \right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n-1}{1+3n} \right)^{2n}} \\ &= \lim_{n\to\infty} \left(\frac{4n-1}{1+3n} \right)^2 = \frac{16}{9} \\ &> 1. \end{aligned}$$

It follows from the ratio test that the series diverges.

9. (20 points)

(a) (10 points) Evaluate the following indefinite integral as an infinite series:

$$\int e^{-x^2}\, dx$$

Answer:

Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in (-\infty, \infty)$. So,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Using the interchangibility of the sum and the definite integral, we get

$$\begin{aligned} \int e^{-x^2}\, dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\, dx \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} \right) + C \end{aligned}$$

(b) (10 points) Use your answer from part (a) to approximate the value of the following definite integral so that the error is less than 1/200:

$$\int_0^1 e^{-x^2}\,dx$$

Express the relevant partial sum of the series as a fraction with whole numerator and denominator. (Hint: use the error bound for alternating series.)

Answer:

Using the solution from part (a), we have

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}.$$

This series converges by alternating series test with $a_n = \frac{1}{(2n+1)n!}$. We can use the alternating series error estimate to approximate $\int_0^1 e^{-x^2} dx$ as S_N, the N-th partial sum, with an error bounded by a_{N+1}. So it is enough to find the smallest possible N where $a_{N+1} < 1/200$. This requires to find N such that

$$\frac{1}{(2N+3)(N+1)!} < \frac{1}{200}$$

which is equivalent to finding N such that

$$(2N+3)(N+1)! > 200.$$

Listing some values we see

$$a_0 = 1 \qquad a_1 = -\frac{1}{3} \qquad a_2 = \frac{1}{10} \qquad a_3 = -\frac{1}{42} \qquad a_4 = \frac{1}{216}$$

Hence $N = 3$ will do and we get

$$\begin{aligned} S_3 = a_0 - a_1 + a_2 - a_3 &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \\ &= \frac{210 - 70 + 21 - 5}{210} = \frac{156}{210} \end{aligned}$$

and

$$\int_0^1 e^{-x^3} dx \approx S_3 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{156}{210} \approx 0.743$$

with an error smaller than $\frac{1}{200} = 0.005$.

10. (20 points)

Calculate the Taylor series for the following functions using any technique that you choose. Show your work. You should have a single sum in your answers. You do not need to specify the radius/interval of convergence.

(a) (10 points) $f(x) = \dfrac{1}{1-2x} - \dfrac{1}{1+2x}$ centered at $x = 0 \quad (a = 0)$.

Answer:

We start by simplifying the expression for $f(x)$ by taking a common denominator.

$$f(x) = \frac{1}{1-2x} - \frac{1}{1+2x} = \frac{4x}{1-4x^2}$$

Recalling that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

we can write

$$f(x) = \frac{4x}{1-4x^2} = 4x \sum_{n=0}^{\infty} 4^n x^{2n} = \sum_{n=0}^{\infty} 4^{n+1} x^{2n+1}.$$

(b) (10 points) $f(x) = e^{-2x}$ centered at $x = 2 \quad (a = 2)$.

Answer:

Taking derivatives, we see that

$$f^{(1)}(x) = -2e^{-2x} \qquad f^{(2)}(x) = 4e^{-2x} \qquad f^{(3)}(x) = -8e^{-2x} \cdots$$
$$f^{(n)}(x) = (-2)^n e^{-2x}.$$

It follows that $f^{(n)}(2) = (-2)^n e^{-4}$ and, using the Taylor series formula,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-2)^n e^{-4}}{n!} (x-2)^n$$

This is scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.

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