Math 162: Calculus IIA

Final Exam ANSWERS

December 16, 2021

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$

$$\pi + \arctan(y/x) \text{for } x < 0$$

$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$

$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$

$$\text{undefined} \text{for } (x, y) = (0, 0)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_{a}^{b} \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The mth Taylor polynomial is

$$T_m(x) = \sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!} (x-a)^n,$$

and the mth Taylor remainder is

$$R_m(x) = f(x) - T_m(x)$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_m(x)| \leq \frac{(x-a)^{n+1} M}{(n+1)!}.$$

Part A

1. (20 points)

(a) (10 points) The form of the partial fraction decomposition of the function is given below:

$$\frac{3x^2+3x+3}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}.$$

Find the coefficients A, B and C.

Answer:

$$\frac{3x^2+3x+3}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$$
$$3x^2+3x+3 = A(x^2+4) + (Bx+C)(x-3).$$

Plug in $x = 3:$

$27+9+3 = 39 = 13A, \Rightarrow A = 3.$

Therefore, $3x^2+3x+3 = 3(x^2+4)+(Bx+C)(x-3) = (3+B)x^2+(C-3B)x+12-3C.$

Matching coefficients of appropriate degrees of x, we get $3 = 3+B$ and $3 = 12-3C.$

So $B = 0$ and $C = 3.$

(b) (10 points) Evaluate the following integral:

$$\int \frac{3x^2+3x+3}{(x-3)(x^2+4)}dx.$$

Answer:

From part (a):

$$\int \frac{3x^2+3x+3}{(x-3)(x^2+4)}dx$$
$$= \int \frac{3}{x-3} + \frac{3}{x^2+4}dx.$$

$\int \frac{3}{x-3}dx = 3\ln|x-3| + C.$

For $\int \frac{3}{x^2+4}dx$ we will use trig substitution:

$\tan(\theta) = x/2 \Rightarrow 2\tan(\theta) = x \Rightarrow 2\sec^2(\theta)d\theta = dx$. Also, $\cos(\theta) = \frac{1}{\sqrt{x^2+4}}$, so $\cos^2(\theta) = \frac{1}{x^2+4}$. After substituting,

$\int \frac{3}{x^2+4}dx = 3\int \cos^2(\theta)\sec^2(\theta)d\theta = 3\int d\theta = 3\theta + C = 3\arctan(x/2) + C.$

So in total,

$$\int \frac{3}{x-3} + \frac{3}{x^2+4}dx = 3\ln|x-3| + 3\arctan(x/2) + C.$$

2. (20 points) Consider the solid $\mathcal{R}$ formed by revolving the function $y = \sqrt{4-x^2}$ around the x-axis for $0 \le x \le 2$.

(a) (10 points) Compute the volume of $\mathcal{R}$.

Answer:

Using the Disk Method, the volume is

$$\begin{aligned} &\pi \int_0^2 4 - x^2 dx \\ =&\pi \left[4x - \frac{x^3}{3}\right]_0^2 \\ =&\frac{16\pi}{3} \end{aligned}$$

(b) (10 points) Compute the surface area of $\mathcal{R}$.

Answer:

The formula for the surface area is

$$
\begin{aligned}
S &= 2\pi \int_0^2 y\sqrt{1+(y')^2}dx \\
&= 2\pi \int_0^2 \sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}dx \\
&= 2\pi \int_0^2 \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx \\
&= 2\pi \int_0^2 \sqrt{4-x^2}\sqrt{\frac{4-x^2}{4-x^2}+\frac{x^2}{4-x^2}}dx \\
&= 2\pi \int_0^2 \sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}dx \\
&= 2\pi \int_0^2 2dx = 2\pi\left[2x\right]_0^2 = 8\pi.
\end{aligned}
$$

3. (10 points) Compute the following integral:

$$\int x\ln(x)dx$$

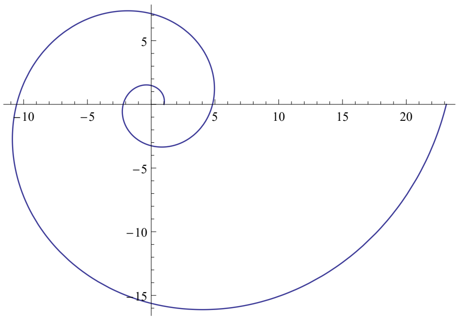
Answer:

We will do integration by parts: $u = \ln(x), dv = xdx, \Rightarrow du = 1/xdx, v = x^2/2$.

$$
\begin{aligned}
\int x\ln(x)dx &= \frac{x^2\ln(x)}{2} - \int \frac{x}{2}dx \\
&= \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C.
\end{aligned}
$$

4. (20 points)

Find the arc length L of the polar curve, $r = e^{\theta/4}$, from $\theta = 0$ to $\theta = 4\pi$.



Answer:

For future reference note that $dr/d\theta = e^{\theta/4}/4$. The polar arc length formula gives

$$\begin{aligned} s &= \int_0^{4\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta \\ &= \int_0^{4\pi} \sqrt{e^{\theta/2} + e^{\theta/2}/16}\, d\theta \\ &= \sqrt{\frac{17}{16}} \int_0^{4\pi} e^{\theta/4}\, d\theta = \frac{\sqrt{17}}{4}\, 4e^{\theta/4}\Big|_0^{4\pi} \\ &= \sqrt{17}(e^{\pi} - 1). \end{aligned}$$

5. (20 points)

(a) (10 points) Compute the volume of a region bounded by the curves $y = x^5 + 1$, $y = 1$ and $x = 1$ and rotated around the y-axis.

Answer:

Using the shell method we have shells of radius x, thickness dx and height $(x^5 + 1) - 1 = x^5$. Therefore

$$V = \int_0^1 2\pi x \cdot x^5 dx = 2\pi \frac{x^7}{7}\Big|_0^1 = \frac{2\pi}{7}$$

(b) (10 points) Find the volume of the region bounded by $y = x^3$, $y = 0$ and $x = 1$ and rotated around line $x = 2$. Use the shell method.

Answer:

Using the shell method we have shells of radius $(2-x)$, thickness dx and height x^3. Thus the volume is

$$\begin{aligned} V &= \int_0^1 2\pi(2-x)x^3\,dx \\ &= \pi\int_0^1 4x^3 - 2x^4\,dx \\ &= \pi\left(x^4 - \frac{2x^5}{5}\right)\Bigg|_0^1 \\ &= \pi\left(1-\frac{2}{5}\right) = \frac{3\pi}{5} \end{aligned}$$

6. (20 points) Compute

$$\int \frac{x^2}{(1-4x^2)^{3/2}}dx$$

Answer:

Consider the right triangle with hypotenuse 1 and sides $2x$ and $\sqrt{1-4x^2}$. Let θ be the angle opposite $2x$. Then we have

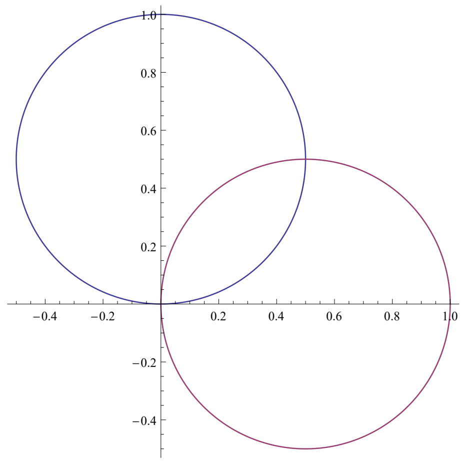
$$x = \frac{\sin\theta}{2} \qquad dx = \frac{\cos\theta\,d\theta}{2} \qquad \sqrt{1-4x^2} = \cos\theta.$$

Writing the integral in term of θ,

$$\begin{aligned} \int \frac{x^2}{(1-4x^2)^{3/2}}dx &= \int \frac{\left(\sin^2\theta/4\right)}{\cos^3\theta}\frac{\cos\theta\,d\theta}{2} \\ &= \frac{1}{8}\int \frac{\sin^2\theta\,d\theta}{\cos^2\theta} \\ &= \frac{1}{8}\int \tan^2\theta\,d\theta \\ &= \frac{1}{8}\int \left(\sec^2\theta - 1\right)\,d\theta \\ &= \frac{1}{8}(\tan\theta - \theta) + c \\ &= \frac{1}{8}\left(\frac{2x}{\sqrt{1-4x^2}} - \arcsin 2x\right) + c \end{aligned}$$

7. (20 points)

(a) (10 points) Find the area of the region both inside the circle $r = \sin\theta$ and outside the circle $r = \cos\theta$ (both equations are in polar coordinates). The two circles are shown below. THEY INTERSECT AT THE ORIGIN AND THE POLAR POINT $(\theta, r) = (\pi/4, \sqrt{2}/2)$.



Answer:

Find the area of the region inside the first circle and outside the second by integrating

$$\int_{\pi/4}^{\pi} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_{\pi/4}^{\pi} \sin^2\theta\,d\theta = \frac{1}{4}\int_{\pi/4}^{\pi} \big(1-\cos 2\theta\big)\,d\theta$$
$$= \frac{1}{4}\left(\frac{3\pi}{4} - \frac{\sin 2\pi}{2} + \frac{\sin(\pi/2)}{2}\right) = \frac{1}{4}\left(\frac{3\pi}{4} + \frac{1}{2}\right) = \frac{3\pi}{16} + \frac{1}{8}$$

and subtracting

$$\int_{\pi/4}^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/2} \big(1+\cos 2\theta\big)\,d\theta$$
$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{\sin\pi}{2} - \frac{\sin(\pi/2)}{2}\right) = \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{\pi}{16} - \frac{1}{8}.$$

So the area of the region is

$$\left(\frac{3\pi}{16} + \frac{1}{8}\right) - \left(\frac{\pi}{16} - \frac{1}{8}\right) = \frac{\pi}{8} + \frac{1}{4}.$$

(b) (10 points) Compute the equation (in Cartesian coordinates x, y) of the tangent line to the circle $r = \sin\theta$ at the points where it intersects the circle $r = \cos\theta$

Answer:

There are two intersection points, at $(x, y) = (0, 0)$ and $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$, at which the tangent lines to the upper circle are horizontal with equation $y = 0$ and vertical with equation $x = \sqrt{2}/2$ respectively.

Part B

8. (20 points) Let q be a positive (greater than 0) real number.

(a)(10 points)

Find the radius of convergence of the series $\sum_{n=0}^{\infty} q^{2n}(x - \pi)^n$.

Answer:

Applying the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{q^{2(n+1)}(x-\pi)^{n+1}}{q^{2n}(x-\pi)^n}\right| = \lim_{n\to\infty} q^2|x-\pi| = q^2|x-\pi|$$

As $q^2|x - \pi| < 1$ if and only if $|x - \pi| < 1/q^2$, we can conclude that the radius of convergence is $1/q^2$.

(b)(10 points) Find the interval of convergence of the series $\sum_{n=0}^{\infty} q^{2n}(x - \pi)^n$.

Answer:

To determine the interval of convergence, we plug in $x = \pi \pm 1/q^2$ into the original expression. For $x = \pi + 1/q^2$, the series becomes $\sum_{n=0}^{\infty} q^{2n}(1/q^2)^n = \sum_{n=0}^{\infty} 1$, which diverges; for $x = \pi - 1/q^2$, the series becomes $\sum_{n=0}^{\infty} q^{2n}(-1/q^2)^n = \sum_{n=0}^{\infty}(-1)^n$, which also diverges. Hence, the interval of convergence is $(\pi - 1/q^2, \pi + 1/q^2)$.

9. (20 points)

(a)(10 points) Consider the series $\sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{1}{n^2} + 1}$. Is this series conditionally convergent, absolutely convergent, or divergent? Explain your answer.

Answer:

The series is divergent, since $\lim_{n\to\infty}(-1)^n\sqrt{\frac{1}{n^2} + 1}$ does not exist.

(b)(10 points) The series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n}$ converges conditionally. How many terms do you need to estimate the sum with an accuracy of $1/1000$?

[The series actually diverges. IGNORE THIS PROBLEM.]

Answer:

10. (20 points)

(a)(10 points) Find a power series expansion of $f(x) = \dfrac{1}{x}$ centered at $x = 1$.

Answer:

Since $\dfrac{1}{x} = \dfrac{1}{1+(x-1)} = \dfrac{1}{1-(-(x-1))}$, we can use the geometric series expansion to get

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

(b)(10 points) Use your series from (a) to find a power series expansion of $\dfrac{1}{x^2}$ centered at $x = 1$.

Answer:

As $\dfrac{1}{x^2} = -\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$, we can differentiate our series from (a) to get

$$\frac{1}{x^2} = -\sum_{n=1}^{\infty} n(-1)^n (x-1)^{n-1} = \sum_{n=1}^{\infty} n(-1)^{n-1}(x-1)^{n-1}.$$

11. (20 points)

(a) (10 points) Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 5^n}$$

Answer:

For all $n \geq 1$, $\frac{1}{n \cdot 5^n} \leq \frac{1}{5^n}$. $\sum\limits_{n=1}^{\infty} \frac{1}{5^n}$ converges because it is a geometric series. Therefore, by the Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{1}{n \cdot 5^n}$ also converges.

One could also use the ratio test, for which the relevant limit is

$$\begin{aligned} L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1/(n+1)5^{n+1}}{1/n \cdot 5^n} \\ &= \lim_{n\to\infty} \frac{n}{5(n+1)} = \frac{1}{5} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{5} \end{aligned}$$

Since $|L| < 1$. the series converges.

(b) **(5 Points)** Find the Maclaurin power series representation for $-\ln|1-x|$. (Hint: What is the Maclaurin series for $1/(1-x)$?)

Answer:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Taking the antiderivative of both sides, we get

$$-\ln|1-x| = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(c) **(5 Points)** Find the value of the sereis of (a) in terms of the natural logarithm.

Answer:

Plug in $\frac{1}{5}$ for x in the Maclaurin series for $-\ln|1-x|$ to get

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 5^n} = -\ln(4/5) = \ln(5/4).$$

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