Math 162: Calculus IIA

Final Exam, Sunday Edition

December 13, 2020

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Write the name of your proctor here.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

Instructions

- **You may not consult the textbook, your notes, the internet, your classmates, friends or any other external source of information. YOUR WEBCAM MUST BE ON AT ALL TIMES.**
- **If you have access to a printer, you may print this exam and write your answers in the spaces provided. Otherwise, write the answers to each problem on a separate sheet of paper. YOU MUST ALSO WRITE AND SIGN THE PLEDGE OF HONESTY AND GIVE ALL OF THE INFORMATION REQUESTED ABOVE.**
- **Show your work and justify your answers. You may use the formulas on the next page. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **You must finish work on this exam by 9:15, and then scan and upload it to Gradescope as previously instructed by 9:30. Exams received after that time will be subject to a penalty.**

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \dfrac{du}{u}$
- $\sec(\theta) = \dfrac{u^2 + 1}{2u}$
- $\tan(\theta) = \dfrac{u^2 - 1}{2u}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Polar coordinate formulas:

- Area:

$$\frac{1}{2}\int r^2 d\theta$$

- Arc length:

$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2}dt.$$

Power series formulas:

- MACLAURIN series for $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}.$$
- Maclaurin series for specific functions:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \qquad \sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \text{where} \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$
- TAYLOR series for $f(x)$ about a:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$
- The nth partial sum of the above, also called the nth Taylor polynomial, is
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!},$$
and the nth Taylor remainder is $R_n(x) = f(x) - T_n(x)$. Taylor's inequality says that
$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$
when x is in an interval centered at a in which $|f^{(n+1)}| \leq M$.

1. (25 points) Consider the region $\mathcal{R}$ under $y = \sin(x)$, for x in $[0, \pi]$.

(a) Compute the volume of the solid formed by revolving $\mathcal{R}$ about the x-axis.
Solution: We will use the disk method:

$$\begin{aligned}
&\int_0^\pi \pi \sin^2(x) dx \\
=&\int_0^\pi \pi \frac{1-\cos(2x)}{2} dx \\
=&\frac{\pi}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi \\
=&\frac{\pi^2}{2}.
\end{aligned}$$

(b) Compute the volume of the solid formed by revolving $\mathcal{R}$ about the y-axis.
Solution: We will use the shell method:

$$\int_0^\pi 2\pi x \sin(x) dx$$

To integrate this, we'll use integration by parts: $u = x, dv = \sin(x)dx$. This makes $du = dx, v = -\cos(x)$. The integral equals

$$\begin{aligned}
&2\pi \left[[-x\cos(x)]_0^\pi + \int_0^\pi \cos(x) dx \right] \\
=&2\pi \left[[-x\cos(x)]_0^\pi + [\sin(x)]_0^\pi \right] \\
=&2\pi [\pi + 0] \\
=&2\pi^2.
\end{aligned}$$

2. (25 points)

Compute the following integral:

$$\int \frac{x}{(x-1)^2(x+1)} dx$$

ANSWER:

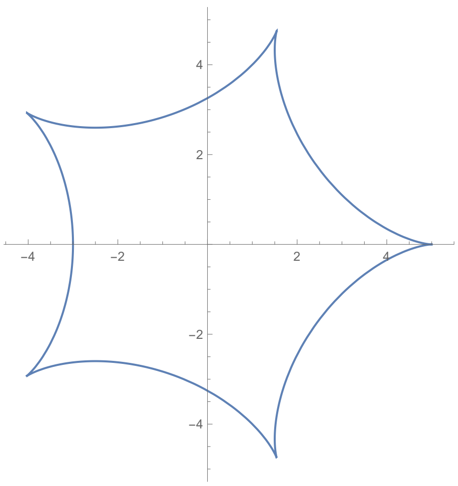
3. (25 points) Find the arc-length of the parametric curve

$$x = 4\cos t + \cos 4t\,,\ \ y = 4\sin t - \sin 4t\,,\ \ 0 \le t \le 2\pi\,.$$

by doing it for $0 \le t \le 2\pi/5$ and multiplying your answer by 5.

You may want to use the trig identities $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin^2\theta = (1-\cos 2\theta)/2$.

The curve for $0 \le t \le 2\pi$ is pictured below.



ANSWER:

4. (25 points)

Let $f : (0, \infty) \to \mathbb{R}$ be a positive, increasing function that is bounded above by a constant M, i.e.

$$f(x) < M \text{ for all } x.$$

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + f(n)}$$

ANSWER:

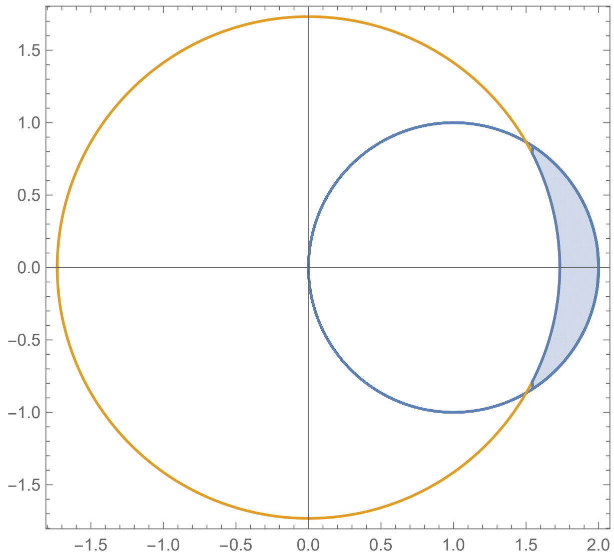
5. (25 points) Let $k \geq 2$ be an integer. Determine whether the series is absolutely convergent, conditonally convergent, or divergent.

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^k}.$$

ANSWER:

6. (25 points)

(a) (15 points) Find the area inside the polar curve $r = 2\cos(\theta)$ and outside the polar curve $r = \sqrt{3}$, as shown below.



ANSWER:

(b) (10 points) Find the arc length of the boundary of the region of part (a).

ANSWER:

7. (25 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{2^n n(\ln n)^3}\,.$$

ANSWER:

8. (25 points)

(a) (15 points) Find the Taylor series centered at 0 of the function $\ln(1 + x^3)$, as well as radius and interval of convergence.

ANSWER:

(b) (10 points) Write the integral

$$\int_0^x \ln(1+t^3)dt$$

as a power series in x.

ANSWER: