Math 162: Calculus IIA

Final Exam, Sunday Edition ANSWERS

December 17, 2020

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \dfrac{du}{u}$
- $\sec(\theta) = \dfrac{u^2+1}{2u}$
- $\tan(\theta) = \dfrac{u^2-1}{2u}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Polar coordinate formulas:

- Area:

$$\frac{1}{2}\int r^2 d\theta$$

- Arc length:

$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2}dt.$$

Power series formulas:

- MACLAURIN series for $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}.$$
- Maclaurin series for specific functions:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \qquad \sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \text{where} \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$
- TAYLOR series for $f(x)$ about a:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$
- The nth partial sum of the above, also called the nth Taylor polynomial, is
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!},$$
and the nth Taylor remainder is $R_n(x) = f(x) - T_n(x)$. Taylor's inequality says that
$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$
when x is in an interval centered at a in which $|f^{(n+1)}| \le M$.

1. (25 points) Consider the region $\mathcal{R}$ under $y = \sin(x)$, for x in $[0, \pi]$.

(a) Compute the volume of the solid formed by revolving $\mathcal{R}$ about the x-axis.
Solution: We will use the disk method:

$$\begin{aligned} &\int_0^\pi \pi \sin^2(x)dx \\ =&\int_0^\pi \pi \frac{1-\cos(2x)}{2}dx \\ =&\frac{\pi}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi \\ =&\frac{\pi^2}{2}. \end{aligned}$$

(b) Compute the volume of the solid formed by revolving $\mathcal{R}$ about the y-axis.
Solution: We will use the shell method:

$$\int_0^\pi 2\pi x \sin(x)dx$$

To integrate this, we'll use integration by parts: $u = x, dv = \sin(x)dx$. This makes $du = dx, v = -\cos(x)$. The integral equals

$$\begin{aligned} &2\pi\left[[-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)dx\right] \\ =&2\pi\left[[-x\cos(x)]_0^\pi + [\sin(x)]_0^\pi\right] \\ =&2\pi\left[\pi + 0\right] \\ =&2\pi^2. \end{aligned}$$

2. (25 points)

Compute the following integral:

$$\int \frac{x}{(x-1)^2(x+1)}dx$$

Answer:

Split the integrand using partial fractions:

$$\frac{x}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}$$

$$\Leftrightarrow x = a(x-1)(x+1) + b(x+1) + c(x-1)^2$$

Using the Heaviside Method: First let $x = 1$:

$$1 = 2b \Rightarrow b = \frac{1}{2}$$

Second let $x = -1$.

$$-1 = c(-2)^2 \Rightarrow c = -\frac{1}{4}$$

To solve for a, we need to match coefficients of x:

$$0x^2 + 1x + 0 = ax^2 - a + \frac{1}{2}x + \frac{1}{2} - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4}$$

Matching the x^2 coefficients, we get that $0x^2 = ax^2 - \frac{1}{4}x^2$, so $a = \frac{1}{4}$.

Putting this together,

$$\begin{aligned}\int \frac{x}{(x-1)^2(x+1)}dx =& \int \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)}dx \\ =& \frac{1}{4}\ln|x-1| + \frac{-1}{2(x-1)} - \frac{1}{4}\ln|x+1| + C.\end{aligned}$$

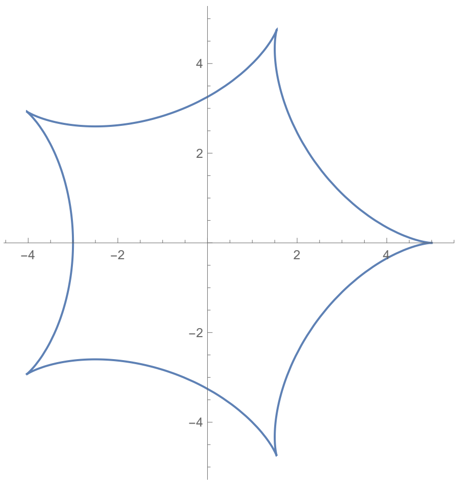
3. (25 points) Find the arc-length of the parametric curve

$$x = 4\cos t + \cos 4t\,,\ y = 4\sin t - \sin 4t\,,\ 0 \le t \le 2\pi\,.$$

by doing it for $0 \le t \le 2\pi/5$ and multiplying your answer by 5.

YOU MAY WANT TO USE THE TRIG IDENTITIES $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ AND $\sin^2\theta = (1-\cos 2\theta)/2$.

The curve for $0 \le t \le 2\pi$ is pictured below.



Answer:

We have

$$dx/dt = -4(\sin t + \sin 4t) \qquad \text{and} \qquad dy/dt = 4(\cos t - \cos 4t).$$

Therefore

$$\begin{aligned}(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\ &= 16(\sin t + \sin 2t)^2 + 16(\cos t - \cos 4t)^2 \\ &= 16(\sin^2 t + 2\sin t\sin 4t + \sin^2 4t + \cos^2 t - 2\cos t\cos 4t + \cos^2 4t) \\ &= 16(2 - 2\cos 5t) = 32(1-\cos 5t) \\ &\qquad \text{since } \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= 64\left(\frac{1-\cos 5t}{2}\right) \\ &= 64\sin^2(5t/2),\end{aligned}$$

so

$$\frac{ds}{dt} = 8|\sin(5t/2)|.$$

By the arc length formula, we have

$$L = 5\int_0^{2\pi/5} ds = 40\int_0^{2\pi/5} \sin(5t/2)\, dt$$

$$\begin{aligned} &= 16\int_0^{\pi} \sin u\, du, \qquad \text{where } u = 5t/2, \text{ so } dt = 2du/5 \\ &= -16\cos u\Big|_0^{\pi} = 32. \end{aligned}$$

4. (25 points)

Let $f : (0, \infty) \to \mathbb{R}$ be a positive, increasing function that is bounded above by a constant M, i.e.

$$f(x) < M \text{ for all } x.$$

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + f(n)}$$

Answer:

First we check absolute convergence. Let $a_n = \frac{1}{\sqrt{n}+f(n)}$. We use the limit comparison test with the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n} + f(n)} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \\ &= \lim_{n\to\infty} \frac{1}{1 + \frac{f(n)}{\sqrt{n}}} \end{aligned}$$

As $0 < f(n) < M$ for all $n > 0$, we have $\lim_{n\to\infty} \frac{f(n)}{\sqrt{n}} = 0$, so the above limit is 1. Since $\sum_{n=1}^{\infty} b_n$ diverges (p-series with $0 < p < 1$), the limit comparison implies that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+f(n)}$ diverges as well.

Next we check convergence. Since $f(x)$ is a positive function, we have $0 < \frac{1}{\sqrt{n}+f(n)} < \frac{1}{\sqrt{n}}$, then it is clear that $\lim_{n\to\infty} \frac{1}{\sqrt{n}+f(n)} = 0$. Moreover, since $f(n)$ and $\sqrt{n}$ are both increasing, $\frac{1}{\sqrt{n}+f(n)}$ is decreasing. By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+f(n)}$ is convergent. Combine with above, the series is conditionally convergent.

5. (25 points) Let $k \geq 2$ be an integer. Determine whether the series is absolutely convergent, conditonally convergent, or divergent.

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^k}.$$

Answer:

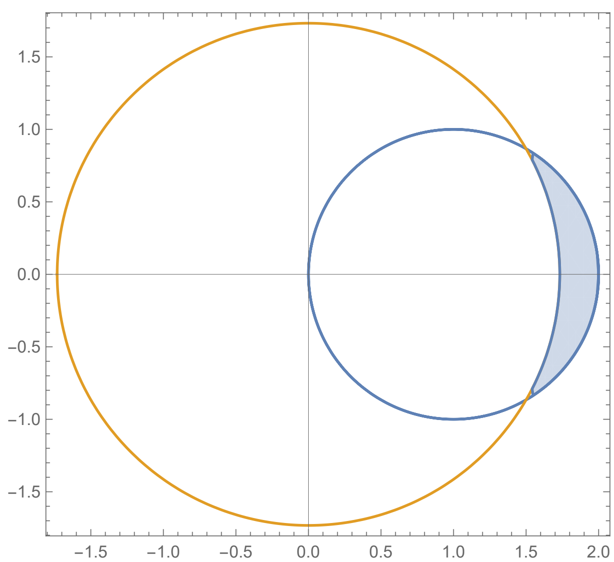
First we test absolute convergence, so we consider the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^k}$. We compute the improper integral

$$\begin{aligned} \int_2^\infty \frac{1}{x(\ln x)^k}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^k}\,dx \\ &= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u^k} \quad \text{where } u=\ln x \\ &= \lim_{t\to\infty}\frac{1}{-k+1}[u^{-k+1}]_{\ln 2}^{\ln t} \\ &= \lim_{t\to\infty}\frac{1}{-k+1}\left(\frac{1}{(\ln t)^{k-1}}-\frac{1}{(\ln 2)^{k-1}}\right) \\ &= \frac{1}{(k-1)(\ln 2)^{k-1}} \quad \text{since } k\geq 2 \end{aligned}$$

Since the improper integral is finite, by the integral test the series is convergent, and that our original series $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^k}$ is absolutely convergent.

6. (25 points)

(a) (15 points) Find the area inside the polar curve $r = 2\cos(\theta)$ and outside the polar curve $r = \sqrt{3}$, as shown below.



Answer:

The curves intersect when $2\cos(\theta) = \sqrt{3}$ or $\cos(\theta) = \frac{\sqrt{3}}{2}$. We know $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$ so the points of intersection are $\theta_1 = -\frac{\pi}{6}$ and $\theta_2 = \frac{\pi}{3}$. Thus,

$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}[(2\cos\theta)^2 - (\sqrt{3})^2]\,d\theta = 2\int_0^{\pi/6} (2\cos^2\theta - 3/2)\,d\theta$$
$$= 2\int_0^{\pi/6} (\cos 2\theta - 1/2)\,d\theta = \sin 2\theta - \theta|_0^{\pi/6} = \sin\pi/3 - \pi/6 = \sqrt{3}/2 - \pi/6.$$

(b) (10 points) Find the arc length of the boundary of the region of part (a).

Answer:

The points of intersection are the same as in part (a). Thus,

$$s = \int_{-\pi/6}^{\pi/6} \sqrt{(\sqrt{3})^2 + 0^2}\,d\theta + \int_{-\pi/6}^{\pi/6} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\,d\theta$$
$$= 2\int_0^{\pi/6} \sqrt{3}\,d\theta + 4\int_0^{\pi/6} d\theta$$
$$= \frac{\pi\sqrt{3}}{3} + \frac{2\pi}{3} = \frac{\pi(2+\sqrt{3})}{3}.$$

7. (25 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{2^n n(\ln n)^3}.$$

Answer:

Solution: Using ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}(n+1)(\ln(n+1))^3}\frac{2^n n(\ln n)^3}{x^n}\right|$$
$$= \left|\frac{x}{2}\right| \cdot \left|\lim_{n\to\infty}\frac{n(\ln(n))^3}{(n+1)(\ln(n+1))^3}\right| = \left|\frac{x}{2}\right|$$

because by the l'Hospital rule

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = 1.$$

To have an absolute convergence series, we need to have $\left|\frac{x}{2}\right| < 1$, so $|x| < 2$ and the radius of convergence is 2. Consider the end points at $x = 2$ and $x = -2$, we have

- $x = -2$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}.$$
By the integral test, the convergence of the series is equivalent to the convergence of the integral
$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx.$$
By letting $\ln x = y$, we can calculate the integral as follows:
$$\int_2^{\infty} \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^{\infty} \frac{dy}{y^3} = \frac{-1}{2y^2}\bigg|_{\ln 2}^{\infty} = \frac{1}{2(\ln 2)^2}.$$
Therefore, the series is also convergent.

- $x = 2$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}.$$
This is absolutely convergent from the case $x = -2$. So, the above series is convergent.

  (Note : You may also use the alternating series test for the convergence.)

8. (25 points)

(a) (15 points) Find the Taylor series centered at 0 of the function $\ln(1 + x^3)$, as well as radius and interval of convergence.

Answer:

The Taylor series of $\ln(1 + x)$ is
$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$
with its radius of convergence 1. Therefore, replacing x by x^3,
$$\ln(1 + x^3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{3n} = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \cdots$$
and its radius of convergence is also 1.

We check the convergence at the end points $x = \pm 1$. When $x = -1$, the series is
$$-\sum_{n=1}^{\infty} \frac{1}{n}$$

and it is diverges because it is the harmonic series. When $x = 1$, we get the alternating harmoninc sereis, which converges by the alternating series test.

Hence the interval of convergence is $(-1, 1]$.

(b) (10 points) Write the integral

$$\int_0^x \ln(1+t^3)dt$$

as a power series in x.

Answer:

$$\begin{aligned}
\int_0^x \ln(1+t^3)dt &= \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^{3n} dt \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^x t^{3n} dt \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{3n+1}}{n(3n+1)} \\
&= \frac{x^4}{4} - \frac{x^7}{2 \cdot 7} + \frac{x^{10}}{3 \cdot 10} - \cdots
\end{aligned}$$

The equation holds for $|x| \leq 1$.