Math 162: Calculus IIA

Final Exam ANSWERS

December 16, 2019

Part A

1. (20 points) Compute the following integral:

$$\int \frac{x+3}{(x-1)^2(x+2)}dx$$

Answer:

The partial fraction expansion we need is

$$\frac{x+3}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$

Multiplying by the denominator, we get

$$x+3 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \tag{1}$$

Substituting $x = 1$ into (1) we get $4 = B \cdot 3$, implying that $B = 4/3$.

Substituting$x = -2$ into (1) we get $1 = C \cdot 9$, implying that $C = 1/9$.

Now, let us expand each term in (1) and sort by degree:

$$x+3 = A(x^2+x-2) + B(x+2) + C(x^2-2x+1)$$
$$x+3 = (A+C)x^2 + (A+B-2C)x - 2A + 2B + C.$$

Consider the coefficients of the x^2 term. On the left hand side, the coefficients of x^2 is 0, so the coefficients on the right hand side should equal 0. Therefore, $0 = A + C$, meaning $A = -1/9$. Hence we have

$$\int \frac{x+3}{(x-1)^2(x+2)}dx = \int \frac{-1}{9(x-1)}dx + \int \frac{4}{3(x-1)^2}dx + \frac{1}{9(x+2)}dx$$
$$= -\frac{1}{9}\ln|x-1| - \frac{4}{3(x-1)} + \frac{1}{9}\ln|x+2| + C.$$

2. (20 points)

Consider the function $y = \sqrt{x+1}$ on the interval $[1, 5]$.

(a) **(10 Points)** Compute the volume of the region bound by the curves $y = \sqrt{x+1}$, $x = 1, x = 5$ and the x-axis, revolved about the x-axis.

Answer:

We can use the Disk (aka Washer) Method:

$$\int_1^5 \pi(\sqrt{x+1})^2 dx = \int_1^5 \pi(x+1)dx$$
$$= \pi\left[\frac{x^2}{2} + x\right]_1^5$$
$$= 16\pi$$

(b) **(10 Points)** Compute the surface area of the region bound by the curves $y = \sqrt{x+1}$, $x = 1$, and $x = 5$, revolved about the x-axis.

Answer:

The derivative of $\sqrt{x+1}$ is $1/2\sqrt{x+1}$. So the formula for the surface area is:

$$\int_1^5 2\pi\sqrt{x+1}\cdot\sqrt{1+\frac{1}{4(x+1)}}dx = 2\pi\int_1^5 \sqrt{x+1+\frac{1}{4}}dx$$
$$= 2\pi\left[\frac{2}{3}(x+\frac{5}{4})^{3/2}\right]_1^5$$
$$= \frac{4}{3}\pi\left(\frac{125}{8} - \frac{27}{8}\right)$$
$$= \frac{49\pi}{3}$$

3. (20 points)

Find the arc length L of the parametric curve, $x = e^t \cos t$, $y = e^t \sin t$, from $t = 0$ to $t = \pi$.

Answer:

$$\begin{aligned} dx/dt &= -e^t \sin t + e^t \cos t = e^t(\cos t - \sin t), \\ dy/dt &= e^t \sin t + e^t \cos t = e^t(\cos t + \sin t), \\ (dx/dt)^2 &= e^{2t}(\cos^2 t + \sin^2 - 2\sin t \cos t) = e^{2t}(1 - 2\sin t \cos t), \\ (dy/dt)^2 &= e^{2t}(\cos^2 t + \sin^2 + 2\sin t \cos t) = e^{2t}(1 + 2\sin t \cos t), \end{aligned}$$

So

$$(dx/dt)^2 + (dy/dt)^2 = 2e^{2t}, \sqrt{(dx/dt)^2 + (dy/dt)^2} \qquad = \sqrt{2}e^t.$$

Hence

$$L = \sqrt{2} \int_0^\pi e^t dt = \sqrt{2}(e^\pi - 1)$$

4. (20 points)

(a) Compute the volume of a region bounded by the curves $y = x^4 + 1$, $y = 1$ and $x = 1$ and rotated around the y-axis.

Answer:

Using the shell method we have shells of radius x, thickness dx and height $(x^4 + 1) - 1 = x^4$. Therefore

$$V = \int_0^1 2\pi x \cdot x^4 dx = 2\pi \frac{x^6}{6}\bigg|_0^1 = \frac{\pi}{3}$$

(b) Set up the integral for the volume of the region bounded by $y = x^3$, $y = 0$ and $x = 2$ and rotated around line $x = 2$. Use the shell method. Do not evaluate the integral.

Answer:

Using the shell method we have shells of radius $(2 - x)$, thickness dx and height x^3. Thus the volume is

$$V = \int_0^2 2\pi(2 - x)x^3 \, dx.$$

5. (10 points)

Evaluate the integral

$$\int \ln(x^{\frac{1}{2}})dx.$$

Answer:

Consider that $\int \ln(x^{\frac{1}{2}})dx = \frac{1}{2}\int \ln x dx$.

Using integration by parts with $u = \ln x$ and $dv = dx$ yields $du = \frac{1}{x}$ and $v = x$, so we have

$$\frac{1}{2}\int \ln x dx = \frac{1}{2}(x \ln x - \int 1dx) = \frac{1}{2}\ln x - \frac{1}{2}x + C.$$

6. (20 points) Compute

$$\int \frac{x^2}{(1-x^2)^{3/2}}dx$$

Answer:

Make the trig substitution $x = \sin\theta$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then we have $dx = \cos\theta d\theta$ and $\sqrt{1-x^2} = \cos\theta$. So

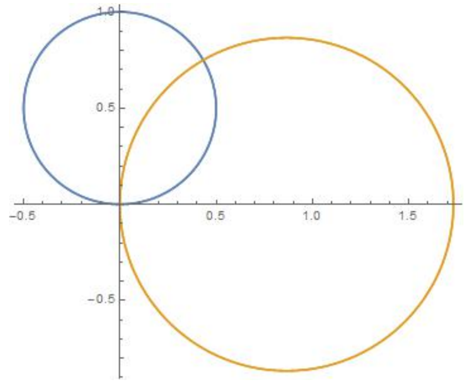
$$\int \frac{x^2}{(1-x^2)^{3/2}}dx = \int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta d\theta = \int \tan^2\theta d\theta = \int \sec^2\theta - 1 d\theta = \tan\theta - \theta + C.$$

We have that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$, so

$$\int \frac{x^2}{(1-x^2)^{3/2}}dx = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C.$$

7. (20 points)

(a) Find the area of the region both inside the circle $r = \sin\theta$ and outside the circle $r = \sqrt{3}\cos\theta$ (both equations are in polar coordinates). The two circles are shown below. THEY INTERSECT AT THE ORIGIN AND THE POLAR POINT $(\theta, r) = (\pi/3, \sqrt{3}/2)$.



Answer:

Find the area of the region inside the first circle and outside the second by integrating:

$$\int_{\pi/3}^{\pi} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2} \int_{\pi/3}^{\pi} \sin^2\theta \, d\theta = \tfrac{1}{4} \int_{\pi/3}^{\pi} \big(1 - \cos 2\theta\big) d\theta = \frac{\pi}{6} + \frac{\sqrt{3}}{16}$$

and subtracting:

$$\int_{\pi/3}^{\pi/2} \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2} \int_{\pi/3}^{\pi/2} 3\cos^2\theta \, d\theta = \tfrac{3}{4} \int_{\pi/3}^{\pi/2} \big(1 + \cos 2\theta\big) d\theta = \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

So the area of the region is $\dfrac{\pi}{24} + \dfrac{\sqrt{3}}{4} \approx 0.563912$.

(b) Compute the equation (in Cartesian coordinates x, y) of the tangent line to the circle $r = \sin\theta$ at the points where it intersects the circle $r = \sqrt{3}\cos\theta$

Answer:

Convert the curve to Cartesian coordinates:

$$x = r\cos\theta = \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$
$$y = r\sin\theta = \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Thus:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan(2\theta)$$

So at the points of intersection $\theta = 0$ and $\theta = \pi/3$:

$$\frac{dy}{dx} = \tan(0) = 0 \qquad \frac{dy}{dx} = \tan(2\pi/3) = -\sqrt{3}$$

Since $(r, \theta) = (\sqrt{3}/2, \pi/2)$ corresponds to $(x, y) = (\sqrt{3}/4, 3/4)$ (scale the 1-2-$\sqrt{3}$ triangle by $\sqrt{3}/4$), the equations of the tangents at those points are:

$$y = 0 \qquad\qquad y - \frac{\sqrt{3}}{2} = -\sqrt{3}\big(x - \tfrac{\sqrt{3}}{4}\big)$$

Part B

8. (20 points)

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{\pi^n (x-2)^n}{\ln n}.$$

Answer:

We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\pi^{n+1}|x-2|^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{\pi^n |x-2|^n} \\ &= \lim_{n\to\infty} \pi \cdot \frac{\ln n}{\ln(n+1)} \cdot |x-2| = \pi|x-2|.\end{aligned}$$

From

$$\pi|x-2| < 1 \Leftrightarrow |x-2| < \frac{1}{\pi},$$

the radius of convergence $R = 1/\pi$.

Now consider the boundary case $x = 2 - 1/\pi$ or $x = 2 + 1/\pi$. Plugging $x = 2 - 1/\pi$ in original series expression, we get

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n},$$

which converges by the alternating series test.

Plugging $x = 2 + 1/\pi$ in original series expression, we get

$$\sum_{n=2}^{\infty} \frac{1}{\ln n},$$

which diverges by the comparison test with

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

So the interval of convergence is $[2 - 1/\pi, 2 + 1/\pi)$.

9. (20 points)

(a) Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Absolutely convergent, conditionally convergent or divergent? Justify your answer.

Answer:

The series obeys all the hypotheses of the Alternating Series Test. Therefore it is convergent.

The corresponding series of positive terms is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n)}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

Which is the divergent 1/2-series. Therefore the series is not absolutely convergent.

Hence the series is conditionally convergent.

(b) The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

converges absolutely. How many terms do you have to add to estimate the sum with an accuracy of 1/100?

Answer:

Solution: The series obeys all the hypotheses of the alternating series test. Therefore, we know the the n'th partial sum s_n will approximate the sum of the series to within the absolute value $|a_{n+1}|$ of the $n+1$'st term. Therefore we need to add n terms, where $|a_{n+1}| < \frac{1}{100}$; i.e., where $\frac{1}{(n+1)^2} < \frac{1}{100}$; i.e., where $(n+1)^2 > 100$; or, taking square roots, where $n + 1 > 10$, i.e., $n > 9$.

So we must add 10 terms to achieve accuracy to within 1/100.

10. (20 points)

(a) Find a power series expansion for the function $f(x) = x^2 e^{-x^2}$ centered at $x = 0$.

Answer:

The power series expansion for e^x centered at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Plugging in $-x^2$ for x we get that

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

Finally,

$$x^2 e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!}/$$

(b) Find the radius of convergence for the series you found in part (a).

Answer:

We use the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{2n+4} n!}{x^{2n+2}(n+1)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2}{(n+1)} \right| \\ &= 0. \end{aligned}$$

So the series converges for all x in $(-\infty, \infty)$.

(c) Compute $f^{(6)}(0)$ and $f^{(2019)}(0)$.

Answer:

$f^{(6)}(0) = 6! \cdot c_6 = (-1)^2 \frac{6!}{2!}$ (Note: use $n = 2$ in the series to find c_6.)

$f^{(2019)}(0) = 2019! \cdot c_{2019} = 2019! \cdot 0 = 0$ (Note: the series has only even powers of x, so all odd-index coefficients are zero.)

11. (20 points)

(a) **(10 Points)** Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$$

Answer:

For all $n \geq 1$, $\frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}$. $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ converges because it is a geometric series. Therefore, by the Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ also converges.

(b) **(5 Points)** Find the Maclaurin power series representation for $-\ln|1-x|$. (Hint: What is the Maclaurin series for $1/(1-x)$?)

Answer:

$\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$. Taking the antiderivative of both sides, we get

$$-\ln|1-x| = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(c) **(5 Points)** What is the value of this series:

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}?$$

Answer:

Plug in $\frac{1}{3}$ for x in the Macluarin series for $-\ln|1-x|$ to get

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} = -\ln(2/3).$$

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