# Math 162: Calculus IIA

Final Exam ANSWERS

December 15, 2015

**Part A**

**1. (15 points)** Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx.$$

**Answer:**

Use the substitution $x = 4\tan\theta$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16(\tan^2\theta+1)} = \sqrt{16\sec^2\theta} = 4\sec\theta.$$

So

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = \int \frac{1}{16\tan^2\theta\,4\sec\theta} 4\sec^2\theta\,d\theta$$
$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{16}\left[-\frac{1}{\sin\theta}\right] + C = -\frac{1}{16\sin\theta} + C.$$

From $\tan\theta = \frac{x}{4}$, by drawing a right triangle with one angle $\theta$, we can check that

$$\sin\theta = \frac{x}{\sqrt{x^2+16}},$$

so the answer becomes

$$-\frac{1}{16}\frac{\sqrt{x^2+16}}{x} + C.$$

**2. (20 points)**

(a) Compute the volume of a region bounded by the curves $y = x^3+1$, $y = 1$ and $x = 1$ and rotated around the $x$-axis.

**Answer:**

Using the washer method we have radii of 1 and $1 + x^3$, so

$$\begin{aligned} V &= \pi \int_0^1 \left((x^3+1)^2 - 1\right)\, dx \\ &= \pi \int_0^1 \left(x^6 + 2x^3\right)\, dx \\ &= \pi \left(\frac{x^7}{7} + \frac{2x^4}{4}\right)\Big|_0^1 \\ &= \pi \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{9}{14}\pi. \end{aligned}$$

(b) Set up the integral for the volume of the region bounded by $y = x^4$, $y = 0$ and $x = 2$ and rotated around the $x$-axis. Use the washer method. Do not evaluate the integral.

**Answer:**

Using the washer method, the radius is $x^4$, si

$$V = \int_0^2 x^8\, dx.$$

**3. (10 points)**

Evaluate the integral

$$\int (\ln x)^2\, dx.$$

**Answer:**

Integrating by parts with $u = (\ln x)^2$ and $dv = dx$, we get $du = 2\frac{\ln x}{x} dx$ and $v = x$, so that the integral becomes

$$\int (\ln x)^2 dx = (\ln x)^2 x - \int 2 \ln x dx$$

Integrating the second integral by parts again we get

$$\int 2 \ln x dx = 2x \ln x - 2 \int x \frac{1}{x} dx = 2x \ln x - 2x + C$$

So that the final answer is

$$\int (\ln x)^2 dx = (\ln x)^2 x - 2x \ln x + 2x + C$$

**4. (20 points)**

(a) Find the partial fraction decomposition of

$$\frac{3x-2}{x^2-x}.$$

**Answer:**

Factoring the denominator we get

$$\frac{3x-2}{x^2-x} = \frac{A}{x} + \frac{B}{x-1}.$$

Multiplying through by denominator we get the equation

$$3x - 2 = A(x-1) + Bx$$

Substitution of $x = 1$, gives $B = 1$. Substitution of $x = 0$, gives $A = 2$. Therefore the partial fraction decomposition is

$$\frac{3x-2}{x^2-x} = \frac{2}{x} + \frac{1}{x-1}$$

(b) Write out the form of the partial fraction decomposition of the function

$$\frac{2+x^3}{x^5+2x^3+x} = \underline{\hspace{8cm}}$$

Do not determine the numerical values of the coefficients.

**Answer:**

We factor the denominator as

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2+1)^2$$

Degree of the denominator is greater than that of the numerator. We obtain

$$\frac{2+x^3}{x^5+2x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

(c) Let

$$f(x) = \frac{1}{x} + \frac{4x+5}{x^2+1}.$$

Evaluate

$$\int f(x)dx.$$

**Answer:**

We split the integral into

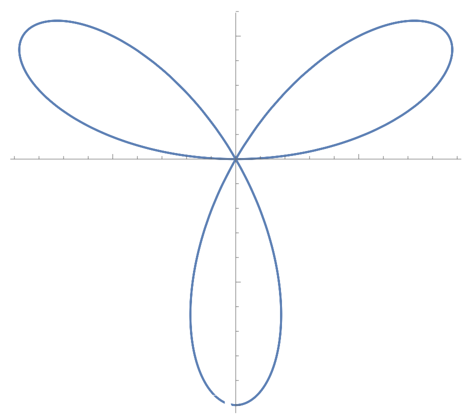
$$\begin{aligned}\int f(x)dx &= \int \frac{1}{x}dx + \int \frac{4xdx}{x^2+1} + \int \frac{5dx}{x^2+1} \\ &= \ln x + \int \frac{4xdx}{x^2+1} + 5\arctan x\end{aligned}$$

For the second integral we use the substitution $u = x^2+1$, so $du = 2xdx$. The second integral becomes

$$\begin{aligned}\int f(x)dx &= \ln x + 5\arctan x + 2\int \frac{du}{u} \\ &= \ln x + 5\arctan x + 2\ln|x^2+1| + C\end{aligned}$$

**5. (15 points)**

Use the polar area formula to find the area of one leaf of the three leafed rose, the polar curve defined by $r = \sin 3\theta$, that is the area for $0 \le \theta \le \pi/3$.



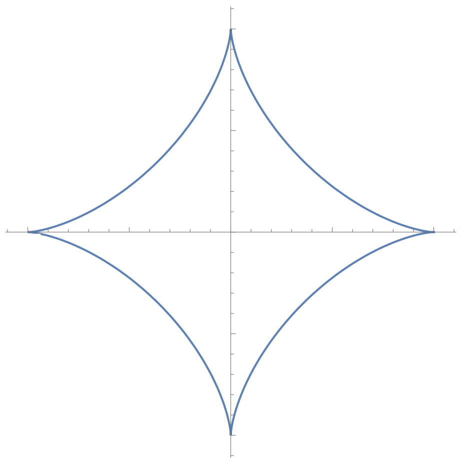
**Answer:**

The area is

$$\begin{aligned}
A &= \int_0^{\pi/3} \frac{r^2}{2} d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta d\theta \\
&= \frac{1}{2}\int_0^{\pi/3} \frac{1-\cos 6\theta}{2} d\theta \qquad \text{by the double angle formula} \\
&= \frac{1}{4}\int_0^{2\pi} \frac{1-\cos u}{6} du \qquad \text{where } u = 6\theta \text{ so } d\theta = du/6 \\
&= \frac{2\pi}{24} = \frac{\pi}{12}.
\end{aligned}$$

**6. (20 points)**

Find the arc length of the astroid, the parametric curve defined by $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \le t \le 2\pi$.



**Answer:**

We will find the arc length for $0 \le t \le \pi/2$ and quadruple it. We have

$$\begin{aligned}
ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt &= \sqrt{(-3\cos^2 t \sin t)^2 + (3\cos t \sin^2 t)^2}\, dt \\
&= 3\cos t \sin t \sqrt{\cos^2 t + \sin^2 t}\, dt \\
&= 3\cos t \sin t\, dt
\end{aligned}$$

so our arc length is

$$\begin{aligned} s &= 12 \int_0^{\pi/2} \cos t \sin t \, dt \\ &= 12 \int_0^1 u, du \qquad \text{where } u = \sin t \text{ and } du = \cos t \, dt \\ &= \left. \frac{12u^2}{2} \right|_0^1 = 6. \end{aligned}$$

**Part B**

**7. (20 points)**

(a) Find a power series representation centered at 1 as well as the radius and interval of convergence for the function

$$f(x) = \frac{2(x-1)}{1+2(x-1)^2}.$$

**Answer:**

$$f(x) = 2(x-1)\frac{1}{1-(-2(x-1)^2)} = 2(x-1)\sum_{n=0}^{\infty}\left(-2(x-1)^2\right)^n = \sum_{n=0}^{\infty}(-1)^n 2^{n+1}(x-1)^{2n+1}$$

for

$$\left|-2(x-1)^2\right| < 1 \Leftrightarrow |x-1| < \frac{1}{\sqrt{2}}.$$

So the radius of convergence $R = \frac{1}{\sqrt{2}}$. Now we consider the boundary cases

$$\left|-2(x-1)^2\right| = 1 \Leftrightarrow x = 1 \pm \frac{1}{\sqrt{2}}.$$

In both cases we have $|a_n| = \sqrt{2}$, and can easily see that the series diverges by the divergence test. We conclude that the interval of convergence is $(1-\sqrt{2}, 1+\sqrt{2})$.

(b) Write the following integral as a power series in $x$. What is the radius of convergence of this power series?

$$\int \frac{2(x-1)}{1+2(x-1)^2} dx$$

**Answer:**

$$\int \frac{2(x-1)}{1+2(x-1)^2}dx = c + \sum_{n=0}^{\infty}(-1)^n 2^{n+1}\int (x-1)^{2n+1}dx = c + \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n+1}(x-1)^{2(n+1)}.$$

for $|x-1| < \frac{1}{\sqrt{2}}$ by the integration theorem. The radius of convergence is $\frac{1}{\sqrt{2}}$ as well.

**8. (20 points)**

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n-2\sqrt{n}+2}$$

**Answer:**

First, consider the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n-2\sqrt{n}+2}\right| = \sum_{n=1}^{\infty}\frac{1}{n-2\sqrt{n}+2}$$

for absolute convergence. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by the $p$-series test and

$$\lim_{n\to\infty}\frac{1}{n-2\sqrt{n}+2}\Big/\frac{1}{n} = \lim_{n\to\infty}\frac{n}{n-2\sqrt{n}+2} = \lim_{n\to\infty}\frac{1}{1-\frac{2}{\sqrt{n}}+\frac{2}{n}} = 1 > 0,$$

by the limit comparison test, the series diverges.

Now, we consider the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n-2\sqrt{n}+2}$$

for conditional convergence. It is an alternating series satisfying

$$\lim_{n\to\infty}\frac{1}{n-2\sqrt{n}+2} = \lim_{n\to\infty}\frac{1}{n(1-\frac{2}{\sqrt{n}}+\frac{2}{n})} = 0.$$

Since

$$n-2\sqrt{n}+2 = (\sqrt{n}-1)^2+1$$

is an increasing function of $n$, $\frac{1}{n-2\sqrt{n}+2}$ is a decreasing function of $n$. So by the Alternating Series test, the series is a conditionally convergent series.

**9. (20 points)**

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{4^n(n+1)}.$$

**Answer:**

**Solution:** We use the ratio test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= |a_{n+1}| \cdot \left|\frac{1}{a_n}\right| = \frac{|x|^{n+1}}{4^{n+1}(n+2)} \cdot \frac{4^n(n+1)}{|x|^n} \\ &= \frac{1}{4} \cdot \frac{n+1}{n+2} \cdot |x| \to \frac{1}{4}|x|\end{aligned}$$

as $n \to \infty$. From

$$\frac{1}{4}|x| < 1 \Leftrightarrow |x| < 4,$$

the radius of convergence $R = 4$.

Now consider the boundary case $x = \pm 4$. Plugging $x = 4$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by the alternating series test.

Plugging $x = -4$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{1}{n+1},$$

which diverges as since this is the harmonic series (without the first term). So the interval of convergence is $(-4, 4]$.

**10. (20 points)**

(a) Find the Taylor series centered at 0 of the function

$$g(x) = \tan^{-1}(x^2) - x^2,$$

as well as the radius of convergence.

**Answer:**

(b) Write the derivative of $g(x)$ as a power series and use it to calculate

$$\left.\frac{dg(x)}{dx}\right|_{x=0}$$

**Answer:**

(a) The Taylor series of $\tan^{-1} x$ is

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots ,$$

which has radius of converges $R = 1$. Therefore, replacing $x$ by $x^2$ gives,

$$\tan^{-1}(x^2) - x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} - x^2 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} = -\frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} \cdots ,$$

which has radius of converges $R = 1$.

(b)

$$\frac{dg(x)}{dx} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \frac{d}{dx} x^{4n+2} = \sum_{n=1}^{\infty} \frac{(-1)^n (4n+2)}{2n+1} x^{4n+1}$$
$$= \sum_{n=1}^{\infty} 2(-1)^n x^{4n+1} = -2x^5 + 2x^9 - 2x^{13} \cdots .$$

The equation holds for $|x| < 1$. It follows that

$$\left.\frac{dg(x)}{dx}\right|_{x=0} = 0.$$

**11. (20 points)**

(a) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$$

is absolutely convergent, conditionally convergent, or divergent.

Hint: You may use the fact that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

**Answer:**

We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{(n+1)}}\cdot\frac{n^n}{n!}\\
&= \lim_{n\to\infty}\frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{(n+1)}}\\
&= \lim_{n\to\infty}(n+1)\frac{n^n}{(n+1)^n(n+1)}\\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\\
&= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \frac{1}{e} < 1.
\end{aligned}$$

so the series is absolutely convergent.

(b) Estimate the sum of the series with an accuracy of $\frac{1}{10}$.

**Answer:**

$\sum_{n=1}^{\infty}(-1)^n\frac{n!}{n^n}$ is an alternating series satisfying

$$0 \leq \frac{n!}{n^n} = \frac{1\cdot 2\cdots n}{n\cdot n\cdots n} \leq \frac{1}{n} \to 0$$

as $n\to\infty$. In a), we showed that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^n < 1.$$

So we can use the error estimate of the Alternating Series Test. From

$$\frac{4!}{4^4} = \frac{1\cdot 2\cdot 3\cdot 4}{4^4} = \frac{6}{64} < \frac{1}{10},$$

the approximate sum is

$$S_3 = \sum_{n=1}^{3}(-1)^n\frac{n!}{n^n} = -1 + \frac{1}{2} - \frac{2}{9} = -\frac{11}{18}.$$