Math 162: Calculus IIA

Final Exam ANSWERS

December 16, 2014

Part A

1. (20 points) Evaluate the integral

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

Solution: (a) The simplest approach is to set $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$ and

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta.$$

So

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{2\cos\theta} 2\cos\theta\,d\theta = \int 4\sin^2\theta\,d\theta = \int 2\left(1-\cos(2\theta)\right)\,d\theta = 2\theta - \sin(2\theta) + C$$

To convert back to x, we use the fact that $\sin(2\theta) = 2\sin\theta\cos\theta = \frac{1}{2}x\sqrt{4-x^2}$. So the answer becomes

$$2\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C.$$

2. (15 points)

Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 1$ about the line $y = 2$.

Solution: We use the washer method. First observe that (since $y = \sqrt{x}$ intersects $x = 0$ and $y = 1$ at the points $(0,0)$ and $(0,1)$, respectively) the bounds on x this region are given by $0 \le x \le 1$. Then for each x such that $0 \le x \le 1$, the cross section of this solid with the plane passing through x perpendicular to the x-axis is an annulus ("washer") of interior radius $r_1(x) = 2 - 1 = 1$ and exterior radius $r_2(x) = 2 - \sqrt{x}$. This washer has area

$A(x) = \pi(r_2(x))^2 - \pi(r_1(x))^2 = \pi((2-\sqrt{x})^2 - 1)$. Therefore this solid has volume given by

$$\begin{aligned} V = \int_0^1 A(x)\,dx &= \pi \int_0^1 ((2-\sqrt{x})^2 - 1)\,dx \\ &= \pi \int_0^1 (3 - 4\sqrt{x} + x)\,dx \\ &= \pi \left[3x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^1 \\ &= \pi(3 - \frac{8}{3} + \frac{1}{2}) = \frac{5\pi}{6}. \end{aligned}$$

3. (15 points) Evaluate the integral

$$\int \sin(x)\cos(x)e^{\sin x}\,dx.$$

Solution: We use the substitution $y = \sin x$. Then $dy = \cos x\,dx$ and the integral becomes

$$\int \sin x \cos x e^{\sin x}\,dx = \int y e^y\,dy.$$

Next we integrate by parts. We let $u = y$ and $dv = e^y dy$, so that $du = dy$ and $v = e^y$. The integral becomes

$$\int y e^y\,dy = y e^y - \int e^y\,dy = y e^y - e^y + C.$$

Changing back to the variable x we conclude that

$$\int \sin x \cos x e^{\sin x}\,dx = \sin x e^{\sin x} - e^{\sin x} + C.$$

4. (20 points)

(a) Find the partial fraction expansion of

$$\frac{1}{x^3 - 4x^2 + 4x}$$

(b) Evaluate the integral

$$\int \frac{1}{x^3 - 4x^2 + 4x}\,dx.$$

(If your answer for part (a) is wrong, you will not receive credit for evaluating the integral of an incorrect function.)

Solution: (a) First we factor $x^3 - 4x^2 + 4x$ as $x(x-2)^2$. So

$$\frac{1}{x^3 - 4x^2 + 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

for some constants A, B, and C. Multiplying through by the denominator gives

$$1 = A(x-2)^2 + Bx(x-2) + Cx.$$

Setting $x = 2$ immediately gives $C = \frac{1}{2}$, and setting $x = 0$ gives $A = \frac{1}{4}$. Setting $x = 1$ gives $1 = A - B + C$, from which it is easy to get $B = -\frac{1}{4}$. So

$$\frac{1}{x^3 - 4x^2 + 4x} = \frac{1/4}{x} + \frac{-1/4}{x-2} + \frac{1/2}{(x-2)^2}.$$

(b) Integrating the answer from (a) gives

$$\frac{1}{4}\ln|x| - \frac{1}{4}\ln|x-2| - \frac{1/2}{x-2}.$$

5. (15 points)

Find the arc length of the parametric curve $x(t) = e^t \cos t$, $y(t) = e^t \sin t$ connecting the point $(1, 0)$ to the point $(-e^\pi, 0)$.

Solution: First observe that the points $(1, 0)$ and $(-e^\pi, 0)$ correspond to $t = 0$ and $t = \pi$, respectively. It follows that the arc length of this curve is given by

$$\begin{aligned}
L &= \int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^\pi \sqrt{(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}\,dt \\
&= \int_0^\pi \sqrt{e^{2t}\cos^2 t - e^{2t}\sin t\cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + e^{2t}\sin t \cos t + e^{2t}\sin^2 t}\,dt \\
&= \int_0^\pi \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)}\,dt \\
&= \sqrt{2}\int_0^\pi e^t\,dt = \sqrt{2}[e^t]_0^\pi = \sqrt{2}(e^\pi - 1).
\end{aligned}$$

6. (15 points)

Use the area formula in polar coordinates to find the area of the region that is both inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

Solution: In polar coordinates, the equation of the given circle is $r = 2$ and the equation of the given line is $r\cos\theta = 1$, or $r = \sec\theta$. The circle and line intersect when $\sec\theta = 2$, or

$\cos\theta = \frac{1}{2}$, which happens when θ is $\pi/3$ or $-\pi/3$. By the area formula in polar coordinates, the area is

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(2^2 - \sec^2\theta\right)\, d\theta = \frac{1}{2}\left(4\theta - \tan\theta\right)\Big|_{-\pi/3}^{\pi/3} = \frac{1}{2}(4\pi/3 - \tan(\pi/3) - (-4\pi/3 - \tan(-\pi/3))).$$

So the area is $4\pi/3 - \sqrt{3}$.

Part B

7. (20 points)

(a) Find a power series representation centered at 0 of the function as well as the radius and interval of convergence.

$$f(x) = \frac{x}{2+x^2}$$

(b) Write the following function as a power series in x. What is the radius of convergence of this power series?

$$\frac{d}{dx}\left(\frac{x}{2+x^2}\right)$$

Solution: (a)

$$f(x) = \frac{x}{2}\frac{1}{1+\frac{x^2}{2}} = \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \frac{x}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{n+1}}$$

for

$$\left|-\frac{x^2}{2}\right| < 1 \Leftrightarrow |x| < \sqrt{2}.$$

So the radius of convergence $R = \sqrt{2}$. Now we consider the boundary cases

$$\left|-\frac{x^2}{2}\right| = 1 \Leftrightarrow x = \pm\sqrt{2}.$$

We can easily see that the series diverges by the divergence test. So, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$.

(b)

$$\frac{d}{dx}\left(\frac{x}{2+x^2}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{d}{dx}\frac{x^{2n+1}}{2^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{2^{n+1}}x^{2n}$$

for $|x| < \sqrt{2}$ by the differentiation theorem. The radius of convergence is $\sqrt{2}$ as well.

8. (20 points)

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=3}^{\infty} \frac{2^n(x+3)^n}{2n+1}.$$

Solution: We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = |a_{n+1}| \cdot \left|\frac{1}{a_n}\right| = \frac{2^{n+1}|x+3|^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{2^n|x+3|^n}$$
$$= 2 \cdot \frac{2n+1}{2n+3} \cdot |x+3| \to 2|x+3|$$

as $n \to \infty$. From

$$2|x+3| < 1 \Leftrightarrow |x+3| < \frac{1}{2},$$

the radius of convergence $R = \frac{1}{2}$.

Now consider the boundary case

$$2|x+3| = 1 \Leftrightarrow 2(x+3) = \pm 1 \Leftrightarrow x = -\frac{5}{2}, -\frac{7}{2}$$

Plugging these in the original series expression, we get

$$\sum_{n=3}^{\infty} \frac{(\pm 1)^n}{2n+1},$$

which diverges for $+1$ by limit comparison with $\sum_{n=3}^{\infty} \frac{1}{2n}$ and converges for -1 by the Alternating series test. So the interval of convergence is $[-\frac{7}{2}, -\frac{5}{2})$.

9. (20 points)

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{(1+n^2) \cdot \tan^{-1} n}$$

Solution:

First, consider the series $\sum_{n=1}^{\infty} \frac{n}{(1+n^2)\cdot\tan^{-1} n}$ for absolute convergence. Since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p-series test and

$$\lim_{n\to\infty} \frac{n}{(1+n^2) \cdot \tan^{-1} n} \cdot n = \frac{2}{\pi},$$

by the limit comparison test, the series diverges.

If we consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{(1+n^2)\cdot \tan^{-1} n}$, since $\frac{n}{1+n^2}$ is positive and decreasing to 0 and $\frac{1}{\tan^{-1} n}$ is positive and decreasing, $\frac{n}{(1+n^2)\cdot \tan^{-1} n}$ is positive and decreasing to 0 and by the alternating series test, the series converges.

So, the series is conditionally convergent.

10. (20 points)

(a) Find the Taylor series centered at 0 of the function $\cos\sqrt{|x|}$, as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x \cos\sqrt{|t|}dt$$

as a power series in x.

Solution: (a) The Taylor series of $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots ,$$

which converegs for all x. Therefore, replacing x by $\sqrt{|x|}$ gives,

$$\cos\sqrt{|x|} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{|x|})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots ,$$

which also converges for all x.

(b)

$$\begin{aligned}\int_0^x \cos\sqrt{|t|}dt &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^n}{(2n)!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)(2n)!} dt \\ &= x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 4!} - \cdots = x - \frac{x^2}{4} + \frac{x^3}{72} - \cdots .\end{aligned}$$

The equation holds for all x.

11. (20 points)

(a) Determine whether the series

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

is absolutely convergent, conditionally convergent, or divergent.

(b) Estimate the sum of the series with an accuracy of $\frac{1}{100}$.

Solution: a)Using the ratio test,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(-1)^{n+1}}{(2n+3)(2n+3)!} \cdot \frac{(2n+1)(2n+1)!}{(-1)^n} \right) \\
&= -\frac{2n+1}{(2n+3)(2n+2)(2n+3)} \\
&= 0,
\end{aligned}$$

so the series is absolutely convergent.

b) Since this is an alternating series, we wish to find n such that

$$\left| \frac{(-1)^n}{(2n+1)(2n+1)!} \right| < 100$$

This is false for $n = 0, 1$, but for $n = 2$ we have $\frac{1}{5 \cdot 5!} = \frac{1}{600} < \frac{1}{100}$ Hence the sum

$$s_1 = 1 - \frac{1}{3 \cdot 3!} = 1 - \frac{1}{18} = \frac{17}{18}$$

is accurate to within 1/100.