Math 162: Calculus IIA

Final Exam ANSWERS

December 17, 2013

Part A

1. (20 points)

(a) Find the partial fraction expansion of

$$\frac{1}{x^3 - x^2 - x + 1}.$$

(b) Calculate the integral

$$\int \frac{1}{x^3 - x^2 - x + 1}\, dx.$$

NOTE: The first part of this problem was designed to help you do the second part. If you did the first part incorrectly, you will not get partial credit for "correctly" using the wrong partial fraction expansion to find the integral.

Solution: (a) $x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2$. This means that

$$\frac{1}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators yields

$$1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

Setting $x = 1$ gives $1 = 2C$ or $C = 1/2$, setting $x = -1$ gives $1 = 4A$ or $A = 1/4$, and setting $x = 0$ gives $1 = A - B + C = 3/4 - B$ or $B = -1/4$. Thus

$$\frac{1}{x^3 - x^2 - x + 1} = \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}.$$

(b) From (a) we get

$$\int \frac{1}{x^3-x^2-x+1}\,dx = \int \left(\frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}\right)\,dx$$

$$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x-1| - \frac{1}{2(x-1)} + C$$

2. (20 points) Evaluate the integral

$$\int (x+1)\sqrt{4-x^2}\,dx$$

[Hint: You may find the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ useful.]

Solution: A right triangle with sides $\sqrt{4-x^2}$ and x will have hypotenuse 2. Let θ be the angle opposite the side with length x. Then we have

$$\begin{aligned} x &= 2\sin\theta, \\ dx &= 2\cos\theta\,d\theta, \\ \sqrt{4-x^2} &= 2\cos\theta, \end{aligned}$$

and $\theta = \arcsin(x/2)$. This gives

$$\begin{aligned} \int (x+1)\sqrt{4-x^2}\,dx &= \int (2\sin\theta+1)4\cos^2\theta\,d\theta \\ &= 8\int \sin\theta\cos^2\theta d\theta + 4\int \cos^2\theta d\theta \\ &= -\frac{8}{3}\cos^3\theta + 2\int(\cos 2\theta + 1)d\theta \\ &= -\frac{8}{3}\cos^3\theta + \sin 2\theta + 2\theta + C \\ &= -\frac{8}{3}\cos^3\theta + 2\sin\theta\cos\theta + 2\theta + C \\ &= -\frac{1}{3}(4-x^2)\sqrt{4-x^2} + \frac{1}{2}x\sqrt{4-x^2} + 2\arcsin(x/2) + C \\ &= \left(-\frac{4}{3} + \frac{x}{2} + \frac{x^2}{3}\right)\sqrt{4-x^2} + 2\arcsin(x/2) + C. \end{aligned}$$

3. (15 points) Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^3$, $y = 0$, and $x = 2$ about the line $x = -2$.

Solution: This is a shell method problem. The volume of a typical shell is

$$[\text{circumference}][\text{height}][\text{thickness}] = [2\pi(2 + x)][x^3][\Delta x].$$

Hence the volume of the solid is

$$\begin{aligned} V &= \int_0^2 2\pi(2 + x)x^3\,dx \\ &= 2\pi \int_0^2 (2x^3 + x^4)\,dx \\ &= 2\pi \left[\frac{1}{2}x^4 + \frac{1}{5}x^5\right]\Big|_0^2 \\ &= \frac{144\pi}{5} \end{aligned}$$

4. (15 points) Evaluate the integral

$$\int \cos\sqrt{x}\,dx.$$

Solution: We begin with a substitution. Let $y = \sqrt{x}$, then $dy = \frac{dx}{2\sqrt{x}}$ so that $dx = 2ydy$. This gives

$$\int \cos\sqrt{x}\,dx = 2\int y\cos(y)\,dy.$$

Next we integrate by parts. Let $u = y$ and $dv = \cos(y)dy$. Then $du = dy$ and $v = \sin(y)$ which gives

$$\begin{aligned} \int y\cos(y)\,dy &= uv - \int v\,du = y\sin(y) - \int \sin(y)\,dy = y\sin(y) + \cos(y) + C \\ &= \sqrt{x}\sin\sqrt{x} + \cos\sqrt{x} + C. \end{aligned}$$

Then

$$\int \cos\sqrt{x}\,dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

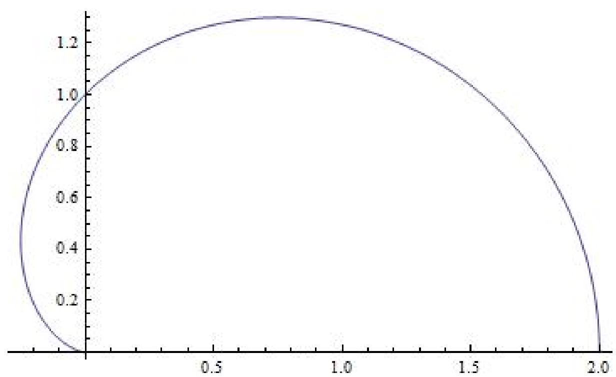
5. (15 points) Find the arc length of the parametric curve $x(t) = e^t + e^{-t}$, $y(t) = 5 - 2t$, connecting the point $(x, y) = (2, 5)$ to the point $(x, y) = (e^3 + e^{-3}, -1)$.

Solution: We begin by observing that $y(t) = 5 - 2t = 5$ when $t = 0$, and $y(t) = -1$ when $t = 3$, so our range of t-values is $0 \leq t \leq 3$. Then the arc length L of this curve is given by

$$\begin{aligned} L &= \int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt \\ &= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt \\ &= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^3 |e^t + e^{-t}|\, dt \\ &= \int_0^3 (e^t + e^{-t})\, dt = [e^t - e^{-t}]_0^3 = e^3 - e^{-3}. \end{aligned}$$

6. (15 points)

The cardioid is the curve defined in polar coordinates by $r = 1 + \cos\theta$. Find the area of the region bounded above by the cardioid and below by the x-axis.



Solution: It is easily verified that the region R bounded above by the cardioid and below by the x-axis is given by

$$R = \{(r, \theta) : 0 \leq r \leq 1 + \cos\theta, 0 \leq \theta \leq \pi\}.$$

We use the formula for area inside a polar curve to compute that the area A of the region R is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^\pi (1 + \cos\theta)^2\, d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\, d\theta \\ &= \frac{1}{2}\int_0^\pi \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi \\ &= \frac{3\pi}{4}. \end{aligned}$$

Part B

7. (20 points)

(a) Find the Maclaurin series expansion of $(\arctan x^2)/x^2$, as well as the interval of convergence.

(b) Find the Maclaurin series for

$$\int_0^x \frac{\arctan t^2}{t^2} dt,$$

as well as the interval of convergence.

Solution: (a) The series for $\arctan x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

which converges for $|x| \leq 1$. Therefore the series for $(\arctan x^2)/x^2$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n+1)} = 1 - \frac{x^4}{3} + \frac{x^8}{5} + \cdots,$$

which also converges for $|x| \leq 1$.

(b) We can find this series by integrating the one above. The result is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)(4n+1)} = x - \frac{x^5}{3 \cdot 5} + \frac{x^9}{5 \cdot 9} + \cdots,$$

which again converges for $|x| \leq 1$.

8. (20 points)

(a) Find the Taylor series centered at 0 of the function $\ln(1-x^2)$, as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x \ln(1-t^2) dt$$

as a power series in x. **Solution:** (a) The Taylor series of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

with its radius of convergence 1. Therefore, replacing x by $-x^2$,

$$\ln(1-x^2) = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \cdots$$

and its radius of convergence is also 1.

We check the convergence at the end points $x = \pm 1$. When $x = 1$, the series is

$$-\sum_{n=1}^{\infty} \frac{1}{n}$$

and it is divergence because it is harmonic series. When $x = -1$, the same thing happens.

(b).

$$\begin{aligned}
\int_0^x \ln(1-t^2)dt &= -\int_0^x \sum_{n=1}^{\infty} \frac{1}{n} t^{2n} dt \\
&= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^x t^{2n} dt \\
&= -\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n(2n+1)} \\
&= -\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{21} - \cdots
\end{aligned}$$

The equation holds for $|x| < 1$.

9. (20 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-1)^n}{3^n \ln n}$$

Solution: Using the ratio test,

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{(-1)^n (x-1)^n} \right|$$

$$\begin{aligned}
&= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \\
&= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{1/n}{1/(n+1)} \text{ by L'Hopital's} \\
&= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{|x-1|}{3} < 1
\end{aligned}$$

So $|x-1| < 3$, and the radius of convergence is 3. For the interval of convergence, we check endpoints.

If $x = -2$,
$$\sum_{n=2}^{\infty} \frac{(-1)^n(-2-1)^n}{3^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}.$$
This series diverges in direct comparison to the series $\sum_{n=2}^{\infty} \frac{1}{n}$, because $1/n < 1/\ln n$, and $\sum \frac{1}{n}$ is divergent (harmonic.)

If $x = 4$,
$$\sum_{n=2}^{\infty} \frac{(-1)^n(4-1)^n}{3^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.$$
This is a convergent alternating series, because $\{\ln n\}$ is an increasing sequence, so $\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$, and because $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.

Hence the interval of convergence is $(-2, 4]$.

10. (20 points)

Determine whether the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n^3-3}}{n^3+1}$$
is absolutely convergent, conditionally convergent, or divergent.

Solution:
$$\begin{aligned}\sum_{n=1}^{\infty} \left|\frac{(-1)^n\sqrt{n^3-3}}{n^3+1}\right| &= \sum_{n=1}^{\infty} \frac{\sqrt{n^3-3}}{n^3+1} \\ &< \sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{n^3} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, ($p = 3/2 > 1$), $\sum_{n=1}^{\infty} \left|\frac{(-1)^n\sqrt{n^3-3}}{n^3+1}\right|$ converges by direct comparison. Then $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n^3-3}}{n^3+1}$ is absolutely convergent.

11. (20 points)

(a) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n(-4)^n}{3^{2n+1}}$$

is absolutely convergent, conditionally convergent, or divergent.

(b) Estimate the sum of the series with an accuracy of $\frac{1}{10}$.

Solution: a.)Using the ratio test,

$\lim_{n\to\infty} \left| \frac{(n+1)(-4)^{n+1}}{3^{2(n+1)+1}} \cdot \frac{3^{2n+1}}{n(-4)^n} \right|$

$$\begin{aligned} &= \frac{4}{9} \lim_{n\to\infty} \frac{n+1}{n} \\ &= \frac{4}{9} < 1 \end{aligned}$$

So the series is absolutely convergent.

b.) Since this is an alternating series, we wish to find n such that

$$\frac{n4^n}{3^{2n+1}} < \frac{1}{10}.$$

This is false for $n = 1, 2$, but

$$\frac{3 \cdot 4^3}{3^7} = \frac{64}{729} < \frac{1}{10}.$$

Hence the sum $s_2 = \frac{-4}{27} + \frac{32}{243} = \frac{-4}{243}$ is accurate to within 1/10.