Math 162: Calculus IIA

Final Exam ANSWERS

December 17, 2011

Part A

1. (20 points)

(a) Find the partial fraction expansion of

$$\frac{1}{x^3 - x^2 - 6x}.$$

(b) Calculate the integral

$$\int \frac{1}{x^3 - x^2 - 6x}\,dx.$$

NOTE: The first part of this problem was designed to help you do the second part. If you did the first part incorrectly, you will not get partial credit for "correctly" using the wrong partial fraction expansion to find the integral.

Solution: (a) $x^3 - x^2 - 6x = x(x+2)(x-3)$. This means that

$$\frac{1}{x^3 - x^2 - 6x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}.$$

Clearing denominators yields

$$1 = A(x+2)(x-3) + B(x)(x-3) + C(x)(x+2).$$

Setting $x = 0$ gives $1 = -6A$ or $A = -1/6$, setting $x = -2$ gives $1 = 10B$ or $B = 1/10$, and setting $x = 3$ gives $1 = 15C$ or $C = 1/15$. Thus

$$\frac{1}{x^3 - x^2 - 6x} = -\frac{1}{6x} + \frac{1}{10(x+2)} + \frac{1}{15(x-3)}.$$

(b) From (a) we get

$$\int \frac{1}{x^3 - x^2 - 6x}\,dx = \int \left(-\frac{1}{6x} + \frac{1}{10(x+2)} + \frac{1}{15(x-3)} \right)\,dx$$

$$= -\frac{1}{6}\int \frac{dx}{x} + \frac{1}{10}\int \frac{dx}{x+2} + \frac{1}{15}\int \frac{dx}{x-3}$$

$$= -\frac{1}{6}\ln|x| + \frac{1}{10}\ln|x+2| + \frac{1}{15}\ln|x-3| + C$$

2. (20 points) Evaluate the integral

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx$$

[Hint: You may find the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ useful.]

Solution: A right triangle with sides $\sqrt{16-x^2}$ and x will have hypotenuse 4. Let θ be the angle opposite the side with length x. Then we have

$$\begin{aligned} x &= 4\sin\theta, \\ dx &= 4\cos\theta\,d\theta, \\ \sqrt{16-x^2} &= 4\cos\theta, \end{aligned}$$

and $\theta = \arcsin(x/4)$. This gives

$$\begin{aligned} \int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int \frac{16\sin^2\theta}{4\cos\theta} 4\cos\theta\,d\theta \\ &= 16\int \sin^2\theta\,d\theta \\ &= 16\int \frac{1}{2}(1-\cos(2\theta))\,d\theta \\ &= 8\theta - 4\sin(2\theta) + C \\ &= 8\theta - 8\sin\theta\cos\theta + C \\ &= 8\arcsin(x/4) - \frac{x\sqrt{16-x^2}}{2} + C \end{aligned}$$

3. (15 points) Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^4$, $y = 0$, and $x = 1$ about the line $x = 2$.

Solution: This is a shell method problem. The volume of a typical shell is

$$[\text{circumference}][\text{height}][\text{thickness}] = [2\pi(2-x)][x^4][\Delta x].$$

Hence the volume of the solid is

$$\begin{aligned} V &= \int_0^1 2\pi(2-x)x^4\,dx \\ &= 2\pi\int_0^1 (2x^4 - x^5)\,dx \\ &= 2\pi\left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right]\Big|_0^1 \\ &= \frac{7\pi}{15} \end{aligned}$$

4. (15 points)

Evaluate the integral

$$\int \cos x \ln(\sin x)\,dx$$

Solution: We use integration by parts. Let $u = \ln(\sin x)$ and $dv = \cos x\,dx$. Then

$$du = \frac{1}{\sin x}\cos x\,dx = \frac{\cos x}{\sin x}\,dx$$

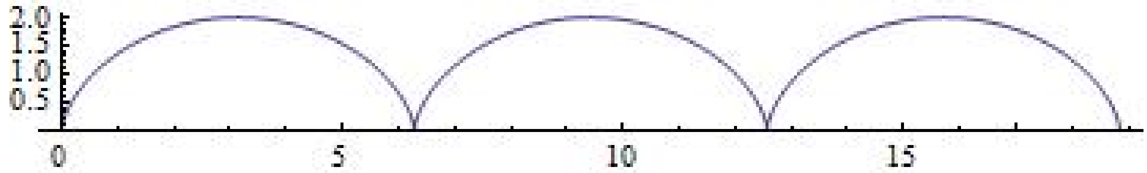
and

$$v = \sin x.$$

Hence

$$\begin{aligned} \int \cos x \ln(\sin x) &= uv - \int v\,du \\ &= \sin x \ln(\sin x) - \int \frac{\cos x}{\sin x}\sin x\,dx \\ &= \sin x \ln(\sin x) - \sin x + C \\ &= \sin x(\ln(\sin x) - 1) + C \end{aligned}$$

5. (20 points) Find the arc-length of the cycloid $x = t - \sin(t)$ and $y = 1 - \cos(t)$, $t \in [0, 6\pi]$. Hint: You may use the identity $\sqrt{(1-\cos t)/2} = |\sin(t/2)|$. It is easier to find the length of one arch and triple it.



Solution: We have

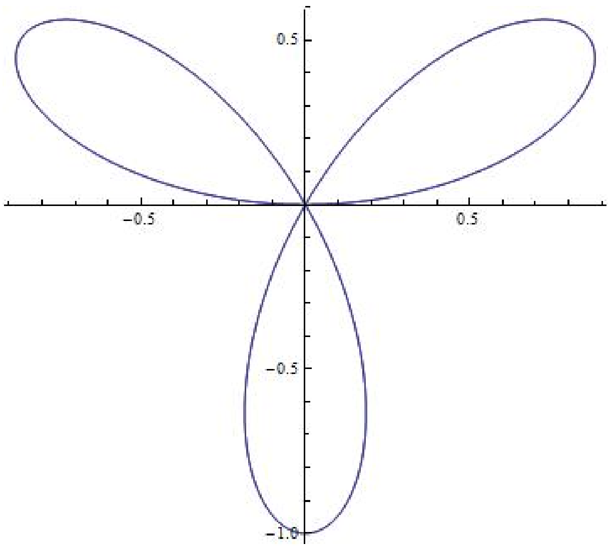
$$\begin{aligned}
\frac{dx}{dt} &= 1 - \cos t \\
\frac{dy}{dt} &= \sin t \\
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
&= (1-\cos t)^2 + \sin^2 t \\
&= 1 - 2\cos t + \cos^2 t + \sin^2 t \\
&= 2\,(1-\cos t) \\
\frac{ds}{dt} &= 2\sqrt{\frac{1-\cos t}{2}} \\
&= |2\sin(t/2)|,
\end{aligned}$$

so the arc length is

$$\begin{aligned}
s &= \int_0^{6\pi} 2\sin(t/2)dt \\
&= 3\int_0^{2\pi} 2\sin(t/2)dt \\
&= 12\int_0^{\pi} \sin u du \qquad \text{where } u = t/2 \\
&= -12\cos u\Big|_0^{\pi} \\
&= 24.
\end{aligned}$$

6. (10 points)

Find the area enclosed by one leaf in the three leafed rose defined in polar coordinates by $r = \sin 3\theta$.



Solution: We get one leaf for $0 \le \theta \le \pi/3$. Hence we have

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{r^2}{2} d\theta \\ &= \int_0^{\pi/3} \frac{\sin^2 3\theta}{2} d\theta \\ &= \int_0^{\pi/3} \frac{1-\cos 6\theta}{4} d\theta \\ &= \frac{1}{24}\int_0^{2\pi} (1-\cos u)\, du \qquad \text{where } u = 6\theta \\ &= \frac{\pi}{12} \end{aligned}$$

Part B
7. (20 points)

(a) Find the Maclaurin series expansion of $(\arctan x)/x$, as well as the interval of convergence.

(b) Find the Maclaurin series for

$$\int_0^x \frac{\arctan t}{t} dt,$$

as well as the interval of convergence.

Solution: (a) The series for $\arctan x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

which converges for $|x| \leq 1$. Therefore the series for $(\arctan x)/x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)} = 1 - \frac{x^2}{3} + \frac{x^4}{5} + \cdots,$$

which also converges for $|x| \leq 1$.

(b) We can find this series by integrating the one above. The result is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2} = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} + \cdots,$$

which again converges for $|x| \leq 1$.

8. (20 points)

(a) Find the Taylor series centered at 0 of the function $\ln(1-x^3)$, as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x \ln(1-t^3)dt$$

as a power series in x. **Solution:** (a) The Taylor series of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

with its radius of convergence 1. Therefore, replacing x by $-x^3$,

$$\ln(1-x^3) = \sum_{n=1}^{\infty} \frac{-1}{n} x^{3n} = -x^3 - \frac{1}{2}x^6 - \frac{1}{3}x^9 - \cdots$$

and its radius of convergence is also 1.

We check the convergence at the end points $x = \pm 1$. When $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{-1}{n}$$

and it is divergence because it is Harmonic series. When $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{-1}{n}(-1)^{3n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots .$$

This is an alternating series with its absolute value $\frac{1}{n}$ decreasing to 0. It converges by the alternating series test.

Therefore, the interval of convergence is $-1 \le x < 1$.

(b).

$$\begin{aligned} \int_0^x \ln(1-t^3)dt &= \int_0^x \sum_{n=1}^{\infty} \frac{-1}{n} t^{3n} dt \\ &= \sum_{n=1}^{\infty} \frac{-1}{n} \int_0^x t^{3n} dt \\ &= \sum_{n=1}^{\infty} \frac{-1}{n(3n+1)} x^{3n+1} \\ &= -\frac{1}{4}x^4 - \frac{1}{14}x^7 - \frac{1}{30}x^{10} - \cdots \end{aligned}$$

The equation holds for $|x| \leqslant 1$.

9. (20 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{2^n n (\ln n)^2} .$$

Solution: Using ratio test, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}(n+1)(\ln(n+1))^2} \frac{2^n n (\ln n)^2}{x^n} \right| \\ &= \left| \frac{x}{2} \right| \cdot \left| \lim_{n\to\infty} \frac{n(\ln(n))^2}{(n+1)(\ln(n+1))^2} \right| = \left| \frac{x}{2} \right| \end{aligned}$$

because by the l'Hospital rule

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = 1.$$

To have an absolute convergence series, we need to have $\left|\frac{x}{2}\right| < 1$, so $|x| < 2$ and the radius of convergence is 2. Consider the end points at $x = 2$ and $x = -2$, we have

- $x = -2$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$
By the integral test, the convergence of the series is equivalent to the convergence of the integral
$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx.$$
By letting $\ln x = y$, we can calculate the integral as follows:
$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{dy}{y^2} = \frac{1}{\ln 2}.$$
Therefore, the series is also convergent.

- $x = 2$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}.$$
This is absolutely convergent from the case $x = -2$. So, the above series is convergent.

(Note : You may also use the alternating series test for the convergence.)

10. (20 points)

Determine whether the series
$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^3}$$
is absolutely convergent, conditionally convergent or divergent.

Solution: Since the denominator of the fraction increases with n, each successive term is closer to 0, so the altenating series test shows that our series converges.

To see whether the series is absolutely convergent, we use the integral test. The series is absolutely convergent if the series $\sum_{n=2}^{\infty} 1/n(\ln n)^3$ is convergent. By the integral test this is the case if and only if the improper integral
$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\, dx$$
is convergent.
$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^3}\, dx &= \int_{\ln(2)}^{\infty} \frac{du}{u^3} \qquad \text{where } u = \ln(x) \text{ so } du = dx/x \\ &= \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2(\ln 2)^2} \right) = \frac{1}{2(\ln 2)^2}.\end{aligned}$$

This is finite, so our series is absolutely convergent.

11. (20 points)

(a) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{(n!)^2}$$

is absolutely convergent, conditionally convergent or divergent.

(b) Estimate the sum of the series within an accuracy of $\frac{1}{36}$.

Solution: (a) By ratio test, let $a_n = \frac{(-2)^n}{(n!)^2}$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{2^n} = \lim_{n\to\infty} \frac{2}{(n+1)^2} = 0 < 1$$

Therefore this series is absolutely convergent.

(b) Let S be the sum of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{(n!)^2} = \sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n = \frac{2^n}{(n!)^2}$, and let S_n be the nth partial sum. The alternating series estimate says that $|S - S_n| < b_{n+1}$. Therefore we need to find the n such that $b_{n+1} \leq \frac{1}{36}$ which guarantees the error is smaller than $\frac{1}{36}$.

$$\frac{2^{n+1}}{((n+1)!)^2} \leq \frac{1}{36}$$

implies that n is at least 3. Thus

$$S \sim -2 + 1 - \frac{2}{9} = -\frac{11}{9}.$$