## Part I

**1.** Compute the following integrals.

(a) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \frac{3x+1}{x^2+x-2}\,dx$

(c) $\displaystyle\int xe^x\,dx$

(d) $\displaystyle\int \sin^2(3x+2)\,dx$

(e) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

**Solutions to Problem 1**

(a) Try substitution $u = \sqrt{x}$ then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and so $du = \frac{dx}{2\sqrt{x}}$ and the integral becomes

$$2\int sin(u)du = -2cos(u) + C.$$

Plugging back in terms of the original variable we get:

$-2cos(\sqrt{x}) + C$ as our answer.

(b) Factor the denominator as $(x+2)(x-1)$ and perform a partial fraction expansion:

$$\frac{3x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Integrating we find:

$$\int \frac{3x+1}{(x+2)(x-1)}dx = Aln(x+2) + Bln(x-1) + C$$

where $C$ is an integration constant.

To solve for $A$ and $B$ we clear denominators in the partial fraction expansion:

$$3x+1 = A(x-1) + B(x+2)$$

Plugging in $x = 1$ we find $4 = 3B$ so $B = \frac{4}{3}$. Plugging in $x = -2$ we find $-5 = -3A$ so $A = \frac{5}{3}$.

Thus our final answer is:

$$\int \frac{3x+1}{(x+2)(x-1)}dx = \frac{5}{3}ln(x+2) + \frac{4}{3}ln(x-1) + C$$

(c) Integration by parts with $u = x$, $dv = e^x dx$. Then $du = dx$ and $v = e^x$. So integral becomes:

$$\int udv = uv - \int vdu = xe^x - \int e^x dx = xe^x - e^x + C$$

where $C$ is an integration constant.

(d) First do a substitution $u = 3x + 2$ then $du = 3dx$ and the integral becomes

$$\frac{1}{3}\int sin^2(u)du = \frac{1}{6}\int [1 - cos(2u)]du = \frac{1}{6}[u - \frac{sin(2u)}{2}] + C$$

where in the first equality above we used a half-angle trig formula. Plugging back for the original variable $x$ we find the answer to be:

$$\frac{1}{6}[3x + 2 - \frac{sin(6x + 4)}{2}] + C$$

(e) Odd number of sines so do a $u = cos(x)$ substitution. Then $du = -sin(x)dx$ and $sin^2(x) = 1 - cos^2(x) = 1 - u^2$ so the integral becomes:

$$\int sin^2(x)cos^2(x)sin(x)dx = -\int (1 - u^2)u^2 du = \int (u^4 - u^2)du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

Plugging back for the original variable $x$ we find the answer is:

$$\frac{cos^5(x)}{5} - \frac{cos^3(x)}{3} + C.$$

**2.**

(a) Find the area of the region enclosed by $y = x^3 - x$ and $y = 3x$.

(b) Find the volume of the solid obtained by rotating the region bounded by $y = 3 + 2x - x^2$ and $y = 3 - x$ about the line $y$-axis.

(c) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

(d) A spring has a natural length of 20cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20cm to 25cm?.

**Solutions to Problem 2**

(a) First solving $3x = x^3 - x$ we get $4x = x^3$ so either $x = 0$ or $4 = x^2$ so curves intersect at three points when $x = 0 (y = 3)$, $x = 2(y = 6)$ and $x = -2(y = -6)$. Drawing a graph shows that between $x = -2$ and $x = 0$ the $y = x^3 - x$ curve is on top and between $x = 0$ and $x = 2$ the line $y = 3x$ is on top. Thus the area enclosed between the curves is in two pieces and is given by:

$$\int_{-2}^{0}(x^3 - x - 3x)dx + \int_{0}^{2}(3x - (x^3 - x))dx = \int_{-2}^{0}(x^3 - 4x)dx + \int_{0}^{2}(4x - x^3)dx$$

Doing the integrals we find the answer is:

$$(\frac{x^4}{4} - 2x^2)|_{-2}^{0} + (2x^2 - \frac{x^4}{4})|_0^2 = (0 - (\frac{16}{4} - 8)) + (8 - \frac{16}{4}) = 8 \text{ unit}^2$$

(b) First note that one curve is a parabola pointing downwards and the other is a line with slope $-1$ and $y$-intercept 3. Solving for intersections we look at $3 + 2x - x^2 = 3 - x$ i.e., $3x = x^2$ which has solutions $x = 0(y = 3)$ and $x = 3(y = 0)$. Drawing a graph, we can see that between $x = 0$ and $x = 3$, the parabola is on top and to the right of the line. Since we rotate about the $y$-axis, we might try $V = \int A(y)dy$. However it is difficult to find $A(y)$ in terms of $y$ in this case. Thus we will use the method of cylindrical shells. Thus

$$V = \int_0^3 (2\pi x)(y_{top} - y_{bottom})dx$$

Here $2\pi x$ represents the circumference of a typical cylindrical shell, $(y_{top} - y_{bottom})$ its height and $dx$ its thickness. Plugging in we get the volume to be:

$$\int_0^3 (2\pi x)(3+2x-x^2-(3-x))dx = 2\pi \int_0^3 (3x^2-x^3)dx = 2\pi(x^3-\frac{x^4}{4})|_0^3 = 2\pi(27-\frac{81}{4}) = \frac{27}{2}\pi \text{ unit}^3$$

(c) We can find intersection points readily as usual at $x = 0(y = 0)$ and $x = 1(y = 1)$. (Solve $x = \sqrt{x}$ by squaring both sides to get $x^2 = x$ etc. ) Graphing, we see that between $x = 0$ and $x = 1$, the curve is a above and to the left of the line. We may evaluate the integral by the method of cylindrical shells again however this time we are rotating around a horizontal line so it will be a $dy$ integral. For a given cylindrical shell at location $0 \leq y \leq 1$, we see the circumference will be given by $2\pi(1 - y)$, the thickness by $dy$ and the height by $x_{right} - x_{left}$.

Thus the volume is given by:

$$V = \int_0^1 2\pi(1-y)(x_{right} - x_{left})dy = \int_0^1 2\pi(1-y)(y-y^2)dy$$

Here for the right curve, we had $y = \sqrt{x}$ so $x = y^2$ etc.

Evaluating the integral we get:

$$V = 2\pi \int_0^1 (y - 2y^2 + y^3)dy = 2\pi[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}]|_0^1 = 2\pi[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}] = \frac{\pi}{6} \text{ unit}^3$$

(d) $F_{spring} = kx$ for an ideal spring where $x$ is the displacement from the natural length and $k$ is the spring constant. From the data given we can find $k$ via: $25 = k(10)$ (note $x$ is displacement from the natural length) so $k = \frac{5}{2}\frac{N}{cm}$. Let $y$ be the location of the endpoint of the spring (with other end at $y = 0$) then the work to stretch the string from $y = 20$ to $y = 25$ is given by:

$$W = \int_{20}^{25} F_{spring}dy = \int_{20}^{25} k(y-20)dy = \int_0^5 kxdx$$

Here we note displacement $x$ from the natural length is given by $x = y - 20$ in the above.

Thus

$$W = k\frac{x^2}{2}|_0^5 = k\frac{25}{2} = \frac{5}{2}\frac{25}{2} = \frac{125}{4}Ncm$$

Since $100cm = 1m$ we can multiply by a conversion factor of $\frac{1m}{100cm}$ to find $W = \frac{125}{400}Nm = \frac{5}{16}Joules$.

**3.** Three improper integrals are given below. Indicate whether they are convergent or divergent and evaluate those which are convergent.

(a) $\displaystyle\int_{-1}^{1} \frac{e^x}{e^x - 1}\, dx$

(b) $\displaystyle\int_{1}^{\infty} \frac{\ln(x)}{x^2}\, dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9 + x^6}\, dx$

**Solutions to Problem 3**

(a) Improper at $x = 0$ so break it up as $\int_{-1}^0 \frac{e^x}{e^x-1}dx + \int_0^1 \frac{e^x}{e^x-1}dx$. Using $u = e^x, du = e^x dx$

we find $\int \frac{e^x}{e^x-1}dx = \int \frac{du}{u-1} = ln(|u-1|)+C = ln(|e^x-1|)+C$. As $x \to 0$, we have $ln(|e^x-1|) \to ln(0) = -\infty$ and so the integrals diverge.
(b) Evaluate as $lim_{t\to\infty} \int_1^t \frac{ln(x)}{x^2}dx$. Use substitution $u = ln(x), x = e^u, du = \frac{1}{x}dx$ to get $\int \frac{ln(x)}{x^2}dx = \int ue^{-u}du = -ue^{-u} - e^{-u} + C = \frac{-ln(x)}{x} - \frac{1}{x} + C$. So we get answer is $lim_{t\to\infty}(-\frac{ln(t)}{t} - \frac{1}{t} + 1)$. By L'hopital's rule, $\frac{ln(t)}{t} \to 0$ as $t \to \infty$ so the integral converges and the value is 1.
(c) By a substitution $u = \frac{x^3}{3}, du = x^2dx$ we get that $9u^2 = x^6$ and $\int \frac{x^2}{9+x^6}dx = \int \frac{1}{9+9u^2}du = \frac{1}{9}arctan(u) + C = \frac{1}{9}arctan(\frac{x^3}{3}) + C$. Thus the improper integral evaluates as $\frac{1}{9}[arctan(\infty) - arctan(-\infty)] = \frac{1}{9}[\frac{\pi}{2} - -\frac{\pi}{2}] = \frac{\pi}{9}$ and converges.

**4.**

(a) Find the length of the curve $y = \ln(x)$, $1 \le x \le \sqrt{3}$.

(b) Rotate the curve $y = \sqrt{x}$, $4 \le x \le 9$ about the $x$-axis. Find the surface area.

(c) Consider the curve given by $x(t) = e^t$, $y(t) = (t-1)^2$. Find the tangent line at time $t = 0$.

(d) Set up an integral giving the length of the curve in (c) from $0 \le t \le 4$.

(e) Set up an integral giving the surface area when the curve in (c) from $0 \le t \le 4$ is rotated about the $y$-axis.

**Solution to Problem 4**

(a) $\int_1^{\sqrt{3}} \sqrt{(\frac{dy}{dx})^2 + 1}dx = \int_1^{\sqrt{3}} \sqrt{\frac{1}{x^2} + 1}dx = \int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}dx$.
There are various ways to do this - none pretty. We do a substitution $x = tan(\theta), dx = sec^2(\theta)d\theta$ and it becomes:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{1+tan^2(\theta)}}{tan(\theta)}sec^2(\theta)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{sec(\theta)}{tan(\theta)}sec^2(\theta)d\theta. = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{sec(\theta)}{tan(\theta)}(1+tan^2(\theta))d\theta$$

This becomes:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} [csc(\theta) + tan(\theta)sec(\theta)]d\theta$$

Since $\frac{d}{d\theta}sec(\theta) = sec(\theta)tan(\theta)$ and $\int csc(\theta)d\theta = \frac{1}{2}ln(\frac{cos(\theta)-1}{cos(\theta)+1})$ we find the answer is

$$[\frac{1}{2}ln(\frac{cos(\theta)-1}{cos(\theta)+1}) + sec(\theta)]|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

(b) $S = \int_4^9 2\pi y\sqrt{(\frac{dy}{dx})^2+1}dx = \int_2^3 2\pi y\sqrt{1+(\frac{dx}{dy})^2}dy$. In this case the second integral seems easier so we use $x = y^2$ so $\frac{dx}{dy} = 2y$ and get

$$S = \int_2^3 2\pi y\sqrt{1+4y^2}dy = \int_{17}^{37} \frac{\pi}{4}\sqrt{u}du = \frac{\pi}{6}u^{\frac{3}{2}}|_{17}^{37} = \frac{\pi}{6}((37)^{3/2} - (17)^{3/2}) \text{ unit}^2.$$

where we used the $u$-substitution $u = 1 + 4y^2$.

(c) $\frac{dy}{dx} = \frac{(dy/dt)}{dx/dt} = \frac{2(t-1)}{e^t}$. At $t = 0$, $\frac{dy}{dx} = -2$ and so tangent line is of the form $y = -2x + b$. Since the curve goes thru $(1, 1)$ at $t = 0$ we find that $1 = -2 + b$ so $b = 3$ and so the tangent line is $y = -2x + 3$.

(d) $\int_0^4 \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}dt = \int_0^4 \sqrt{4(t-1)^2 + e^{2t}}dt$.

(e) $\int_0^4 2\pi x\sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}dt = \int_0^4 2\pi e^t\sqrt{4(t-1)^2 + e^{2t}}dt$.

**5.** Consider the curve given in polar coordinates by the equation $r = 1 + \cos(\theta)$.

(a) Give an accurate sketch of this curve.

(b) Find the area enclosed by one loop of $r = 3\cos(5\theta)$.

**Solution to problem 5**

(a) Cardiod curve (see book for sketch.)

(b) This is a 5-petaled rose that is swept out twice as $\theta$ ranges from 0 to $2\pi$. Thus as $\theta$ ranges from 0 to $\pi$, all five petals are swept out. By symmetry we conclude that the area of one of the five petals is:

$$\frac{1}{5}\int_0^\pi \frac{r^2}{2}d\theta = \frac{9}{10}\int_0^\pi cos^2(5\theta)d\theta = \frac{9}{20}\int_0^\pi [1 + cos(10\theta)]d\theta = \frac{9}{20}[\theta + \frac{sin(10\theta)}{10}]_0^\pi = \frac{9\pi}{20}$$

where we used a half-angle formula.

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**Part II**

**6.** Consider the following geometric series. Find their sum if they converge or write "divergent" otherwise.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

**Solution to problem 6**

(a) Written out, the series is $\frac{1}{4} + \frac{-3}{4^2} + \frac{3^2}{4^3} + \ldots$ which is geometric with $a = \frac{1}{4}$ and $r = \frac{-3}{4}$. Since $|r| < 1$ the series converges to $\frac{a}{1-r} = \frac{\frac{1}{4}}{1+\frac{3}{4}} = \frac{1}{7}$.

(b) Written out, the series is $\frac{1}{5} + \frac{-6}{5^2} + \frac{-6}{5^3} + \ldots$ which is geometric with $a = \frac{1}{5}$ and $r = \frac{-6}{5}$. Since $|r| \geq 1$, the series diverges. (Note: The formula $\frac{a}{1-r}$ is only valid when $|r| < 1$ i.e., when the series converges. Don't use it when $|r| \geq 1$!!!)

(c) Writing it out as usual, we see the series is geometric with $a = e$ and $r = \frac{e}{3}$. Since $|r| < 1$ the series converges to $\frac{a}{1-r} = \frac{e}{\frac{e}{3}} = 3$.

**7.** Determine whether each of the following series is Absolutely Convergent (AC), converges but is not absolutely convergent, i.e. is Conditionally Convergent (CC), or is Divergent (D) and give a short reason why. For example, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ is D by comparison with the Harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ .

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

**Solutions to Problem 7**

(a) Expanding factorials and cancelling common terms we see $a_n = \frac{2^n n!}{(n+2)!} = \frac{2^n}{(n+2)(n+1)}$. As $n \to \infty$, the exponential dominates and $lim_{n\to\infty} a_n = \infty \neq 0$ thus the series diverges as the terms being summed to not head towards zero. (D)

(b) $a_n = \frac{n^2-1}{n^2+1}$. Again as $lim_{n\to\infty} a_n = 1 \neq 0$ we see the series diverges as the terms being summed to not head towards zero. (D)

(c) $a_n = \frac{(-1)^n}{nln(n)^2}$. Note $|a_n| = \frac{1}{nln(n)^2}$. Since $f(x) = \frac{1}{xln(x)^2}$ is a decreasing, positive, continuous function on $[2, \infty)$, the integral test shows that $\sum_{n=2}^{\infty} |a_n|$ converges if and only if $\int_2^{\infty} f(x)dx$ converges. Calculating the integral with a substitution $u = ln(x), du = \frac{1}{x}dx$ we get:

$$\int_2^{\infty} \frac{1}{xln(x)^2} dx = \int_{ln(2)}^{\infty} \frac{1}{u^2} du = \frac{-1}{u} |_{ln(2)}^{\infty} = \frac{1}{ln(2)}$$

Thus the integral converges and hence $\sum_{n=2}^{\infty} |a_n|$ converges and hence $\sum_{n=2}^{\infty} a_n$ converges absolutely. (AC).

(d) Here $a_n = \frac{1}{n^3+3n^2}$. Consider the series $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{1}{n^3}$. Since $lim_{n\to\infty} \frac{a_n}{b_n} = 1$ the limit comparison test shows that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. This latter series converges as it is a $p$-series with $p = 3 > 1$. Thus the answer is (AC) (note series has positive terms).

(e) $a_n = \frac{(-2)^{2n}}{n^n}$. Thus $|a_n|^{\frac{1}{n}} = \frac{|2|^2}{n}$ and $lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0 < 1$. Thus by the root test, the series converges absolutely. (AC)

**8.** Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2 5^n}$

(a) Find the radius of convergence of this power series.

(b) Find the interval of convergence of this power series (be sure to check endpoints).

**Solutions to problem 8** (a) Can use the ratio test or just use the formula $R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$. If we use the formula we have to be careful to take the right $c_n$. In this formula $c_n$ is the coefficient in front of $(x-\alpha)^n$ in general. Thus we have to rewrite our power series as $\sum_{n=1}^{\infty} \frac{3^n}{n^2 5^n}(x - \frac{2}{3})^n$ to see $c_n = \frac{3^n}{n^2 5^n}$ and that the power series is centered at

$\alpha = \frac{2}{3}$. We compute

$$R = lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|} = lim_{n\to\infty}\frac{\frac{3^n}{n^2 5^n}}{\frac{3^{n+1}}{(n+1)^2 5^{n+1}}} = lim_{n\to\infty}\frac{5(n+1)^2}{3n^2} = \frac{5}{3}.$$

(b) The endpoints of the interval of convergence are $\frac{2}{3} - \frac{5}{3} = -1$ and $\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$. At $x = -1$ the series becomes $\Sigma_{n=1}^{\infty}\frac{(-5)^n}{n^2 5^n} = \Sigma_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges in fact absolutely as $\Sigma_{n=1}^{\infty}\frac{1}{n^2}$ converges. (p-series with $p = 2 > 1$.)

On the other hand at $x = \frac{7}{3}$ the series becomes $\Sigma_{n=1}^{\infty}\frac{1}{n^2}$ (after simplification) and this again converges as it is a $p$-series with $p = 2 > 1$. Thus the series actually converges at the two end points of the interval and so the interval of convergence is: $-1 \leq x \leq \frac{7}{3}$.

**9.** Each of the functions below has a Taylor series about $x = 0$. Find the Taylor series.

(a) $\dfrac{\cos(x) - 1}{x^2}$

(b) $\dfrac{x}{1 + x^3}$

(c) $\displaystyle\int \sin(x^2)\,dx$

(d) $\dfrac{d}{dx}xe^{x^3}$

(e) $\ln(1 - x)$

(f) $\arctan(x)$

**Solution to problem 9**

(a) We know $cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ thus

$$cos(x) - 1 = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Thus

$$\frac{cos(x) - 1}{x^2} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n)!} = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots$$

(b) We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

Substituting $-x^3$ for $x$ we get

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{3n} \text{ for } |x^3| < 1$$

Multiplying by $x$ we get:

$$\frac{x}{1+x^3} = x - x^4 + x^7 - x^{10} + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{3n+1} \text{ for } |x| < 1$$

(c) We know that

$$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Replacing $x$ with $x^2$ we get

$$sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

Integrating we get:

$$\int sin(x^2)dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} + C = \frac{x^3}{3} - \frac{x^7}{(7)(3!)} + \frac{x^{11}}{(11)(5!)} - \frac{x^{15}}{(15)(7!)} + \cdots + C$$

where $C$ is an integration constant.

(d) We start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and substitute $x^2$ for $x$ to get

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Multiply by $x$ to get

$$xe^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

Finally we differentiate both sides to get:

$$\frac{d}{dx} xe^{x^2} = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{n!}$$

(e) We start with

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1$$

The we integrate both sides to get

$$-ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \text{ for } |x| < 1$$

where $C$ is an integration constant. Putting in $x = 0$ we see that $C = 0$ as $ln(1) = 0$. Thus we get:

$$ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \text{ for } |x| < 1$$

(f) Start with series for $\frac{1}{1-x}$. Substitute $-x^2$ for $x$ to get the series for $\frac{1}{1+x^2}$. Finally integrate to get the series for $arctan(x)$. Answer is:

$$arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ for } |x| < 1$$

**10.**

(a) Write down the general form of the Taylor series of a function $f(x)$ at $x = a$ (or about $x = a$ or centered at $x = a$).

(b) Write down the Taylor series for $f(x) = \ln(x)$ at $x = 5$. You can either use summation notation or write down the first 5 non-zero terms.

**Solution to Problem 10**

(a) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$.

(c) $f(x) = ln(x) = f^{(0)}(x)$. We compute $f^{(1)}(x) = \frac{1}{x}$, $f^{(2)}(x) = \frac{-1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$, $f^{(4)}(x) = \frac{-6}{x^4}$. Plugging in $x = 5$ we get: $f^{(0)}(5) = ln(5), f^{(1)}(5) = \frac{1}{5}, f^{(2)}(5) = \frac{-1}{25}, f^{(3)}(5) = \frac{2}{125}, f^{(4)}(5) = \frac{-6}{5^4}$. Thus the first 5 terms of the Taylor series are:

$$f(x) = ln(5) + \frac{1}{5}(x-5) + \frac{-1}{(25)2!}(x-5)^2 + \frac{2}{(125)(3!)}(x-5)^3 + \frac{-6}{(5^4)(4!)}(x-5)^4 + \ldots$$