Math 162: Calculus IIA

Fall 2007 Final Exam ANSWERS

December 16, 2013

Part A

1. (10 points)

A circular swimming pool has a diamater of 24 ft., the sides are 5 ft. high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft^3.)

Solution:

Let y be the vertical distance from the top of the pool. In hthis porblem it ranges from 1 to 5. The work required to lift the layer of water at distance y over the top is $W = F * y\Delta y$, where Δy is the thickness of the layer. Then $F = 62 * 5 * V$, where V is the the volume of the layer, which is $\pi * 12^2 * \Delta y$. Now the work required to pump all the water out is given by:

$$\begin{aligned} W &= \int_1^5 \pi \cdot 12^2 \cdot 62.5 \cdot y\, dy \\ &= \pi \cdot 12^2 \cdot 62.5 \cdot \tfrac{y^2}{2}\Big|_1^5 \\ &= \pi \cdot 12^2 \cdot 62.5 \cdot (\tfrac{25}{2} - \tfrac{1}{2}) \\ &= \pi \cdot 12^3 \cdot 62.5\,\text{ft-lb} \\ &= 108,000\,\pi\,\text{ft-lb} \end{aligned}$$

2. (10 points)

Find the definite integral

$$\int_0^{\frac{\pi}{2}} x\cos(2x)\, dx$$

Solution: We use integration by parts with

$$\begin{aligned} u &= x & dv &= \cos(2x)\, dx \\ du &= dx & v &= \tfrac{\sin(2x)}{2} \end{aligned}$$

so

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos(2x)\,dx &= \left.\frac{x\sin(2x)}{2}\right|_0^{\frac{\pi}{2}} - \int_0^{\pi} \frac{\sin(2x)}{2}\,dx \\ &= 0 - \left.\frac{\cos(2x)}{4}\right|_0^{\frac{\pi}{2}} \\ &= \frac{1}{2}\end{aligned}$$

3. (10 points)

Solve this integral:

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

We use the substitution $x = 3\sin\theta$, so that $dx = 3\cos\theta d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Then

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta = \int cot^2\theta d\theta \\
&= \int (\csc^2\theta - 1) d\theta \\
&= -\cot\theta - \theta + C
\end{aligned}$$

Drawing a triangle, we see that $-\cot\theta - \theta + C$ reduces to

$$-\frac{\sqrt{9-x^2}}{x} - \arcsin(x/3) + C$$

4. (10 points)

Evaluate this integral:

$$\int \frac{1}{x^2+x}\,dx$$

We use partial fractions:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Adding the fractions on the right side of the equation and comparing numerators we obtain:

$$1 = A(x+1) + Bx,$$

and it follows that $A = 1$ an $B = -1$. So the integral becomes

$$\int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = (\ln|x| - \ln|x+1|) + C$$

5. (10 points)

(a) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge or diverge? Why?

(b) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converge or diverge? Why?

Solution (a): The function $f(x) = \frac{\ln x}{x}$ is positive, continous and decreasing (look at derivative!) for $x > 1$ Thus we can apply the integral test

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\bigg|_1^{\infty} = \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series is also divergent.

Solution (b): The general term of the series is $a_n = \frac{n^2}{5n^2+4}$. Then:

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

Thus, the series diverges by the Divergence Test.

6. (10 points) Find all x for which the following power series converges, i.e. find the interval of convergence:

$$\sum_{n=1}^{\infty} \frac{-1^n}{n+1}(x+1)^n$$

Solution: We use the Ratio Test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{-1^{n+1}}{n+2}(x+1)^{n+1}}{\frac{-1^n}{n+1}(x+1)^n} \right| &= \lim_{n\to\infty} |x+1| \frac{n+1}{n+2} \\ &= |x+1| \lim_{n\to\infty} \frac{1+1/n}{1+2/n} \\ &= |x+1| \end{aligned}$$

Since the series converges for $|x+1| < 1$, the radius of convergence is 1. We still have to test the endpoints:

When $x = 0$, the series becomes

$$\sum_{n=1}^{\infty} \frac{-1^n}{n+1}$$

which converges by the Alternating Series Test. When $x = 2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{-1^n}{n+1} \cdot (-2)^n$$

This is a divergent series. You can see this by the Divergence Test or by limit comparison to a harmonic test.

Therefore, the interval of convergence for this series is $(-2, 0]$.

Part B
7. (10 points)

The power series for e^{-x^2} is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \cdots \end{aligned}$$

Use this to get the power series for the area under the bell curve,

$$f(x) = \int_0^x e^{-t^2} dt.$$

You can either use summation notation or write down the first 5 non-zero terms.

Solution: Integrating the given series term by term gives

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)} \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} + \cdots \end{aligned}$$

8. (10 points)

(a) Find $T_3(x)$, the third degree Taylor polynomial for $f(x) = \sqrt{x}$ at $a = 4$.

(b) Use Taylor's inequality to find the largest integer k such that the error when $T_3(x)$ is used as an approximation for $f(x)$ on the interval $4 \le x \le 5$ is less than 10^{-k}.

Solution (a): We should first compute four derivatives and their values at $a = 1$:

$$\begin{aligned} f(x) &= x^{1/2} & f(4) &= 2 \\ f'(x) &= \frac{1}{2x^{1/2}} & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{4x^{3/2}} & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{8x^{5/2}} & f'''(4) &= \frac{3}{256} \\ f^{(4)}(x) &= \frac{-15}{16x^{7/2}} & f^{(4)}(4) &= \frac{-15}{2024} \end{aligned}$$

Thus

$$\begin{aligned} T_3(x) &= 2 + \frac{x-4}{4} - \frac{(x-4)^2}{32 \cdot 2!} + \frac{3(x-4)^3}{256 \cdot 3!} \\ &= 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} \end{aligned}$$

Solution (b): Notice that $4 \le x \le 5$ means that $|x-4| \le 1$. Also, since $|f^{(4)}(x)| = 15/16x^{7/2}$, we know that

$$|f^{(4)}(x)| \le \frac{15}{16 \cdot 4^{7/2}} = \frac{15}{1024},$$

on the interval $4 \le x \le 5$. Thus, by Taylor's Inequality, we have:

$$\begin{aligned} |R_3(x)| &\le \tfrac{15}{1024 \cdot 4!}|x-1|^4 \\ &\le \tfrac{15}{24 \cdot 1024} \\ &< 10^{-3} \end{aligned}$$

9. (10 points) (a) Write the general formula for the Taylor series of a function $f(x)$ at a (or "about a" or "centered at a").

(b) Write the Taylor series of $f(x) = e^{2x}$ at $a = 1$. You can either use summation notation or write down the first 5 non-zero terms.

Solution: (a) The general formula is

$$\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

(b) For $f(x) = e^{2x}$, the nth derivitive is $f^{(n)}(x) = 2^n e^{2x}$, so $f^{(n)}(1) = 2^n e^2$ and the series is is

$$\begin{aligned}
\sum_{n=0}^{\infty} f^{(n)}(1)\frac{(x-1)^n}{n!} &= e^2\sum_{n=0}^{\infty} 2^n\frac{(x-1)^n}{n!} \\
&= e^2 + 2e^2(x-1) + \frac{4e^2(x-1)^2}{2!} + \frac{8e^2(x-1)^3}{3!} + \frac{16e^2(x-1)^4}{4!} + \cdots \\
&= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2(x-1)^3}{3} + \frac{2e^2(x-1)^4}{3} + \cdots
\end{aligned}$$

10. (10 points) Consider the cycloid defined by the parametric equations

$$x = 2(t - \sin t) \qquad \text{and} \qquad y = 2(1 - \cos t).$$

for $0 \le t \le 2\pi$.

(a) For which values of t is the tangent line vertical? Find the corresponding points.

(b) For which values of t is the tangent line horizontal? Find the corresponding points.

Solution: We have

$$\frac{dx}{dt} = 2(1 - \cos t) \qquad \text{and} \qquad \frac{dy}{dt} = 2\sin t$$

(a) The tangent line is vertical when $dx/dt = 0$ and $dy/dt \neq 0$, i.e. when $\cos t = 1$,which means $t = 0$ or 2π so $(x, y) = (0, 0)$ or $(4\pi, 0)$.

(b) It is horizontal when $dy/dt = 0$ and $dx/dt \neq 0$, i.e. when $\sin t = 0$ but $\cos t \neq 1$. This happens when $t = \pi$ and $(x, y) = (2\pi, 4)$.

11. (10 points) Find the length of the cycloid of the previous problem for $0 \leq t \leq \pi$.

Hint: Use the half angle formula $\sin(\theta/2) = \sqrt{(1-\cos\theta)/2}$.

Solution:

We have

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= (2(1-\cos t))^2 \\ &= 4 - 8\cos t + 4\cos^2 t \\ \left(\frac{dy}{dt}\right)^2 &= (2\sin t)^2 \\ &= 4\sin^2 y \end{aligned}$$

so

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{4 - 8\cos t + 4\cos^2 t + 4\sin^2 t} \\ &= \sqrt{8 - 8\cos t} \\ &= 4\sqrt{(1-\cos t)/2} \\ &= 4\sin(t/2). \end{aligned}$$

It follows that the arc length is

$$\begin{aligned} L &= 4\int_0^{\pi} \sin(t/2)dt \\ &= 8\int_0^{\pi/2} \sin(u)du \qquad \text{where } u = t/2 \text{ and } dt = 2du \\ &= 8. \end{aligned}$$

12. (10 points)

Find the area of the surface obtained rotating the semicircle $y = \sqrt{25 - x^2}$, $3 \le x \le 4$, about the x-axis.

Solution: Let $f(x) = \sqrt{25 - x^2}$, so

$$\begin{aligned} f'(x) &= \frac{x}{\sqrt{25 - x^2}} \\ 1 + f'(x)^2 &= 1 + \frac{x^2}{25 - x^2} = \frac{25}{25 - x^2} \\ \sqrt{1 + f'(x)^2} &= \frac{5}{\sqrt{25 - x^2}} \end{aligned}$$

Then the surface area is

$$\begin{aligned} S &= \int_3^4 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx \\ &= \int_3^4 2\pi \sqrt{25 - x^2} \frac{5}{\sqrt{25 - x^2}}\, dx \\ &= 10\pi \int_3^4 dx \\ &= 10\pi. \end{aligned}$$

13. (10 points)

Find the area enclosed by the 8-leafed rose defined by $r = \sin 4\theta$ for $0 \le \theta \le 2\pi$.

Solution: Using the area formula for polar curves, we get

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} \sin^2 4\theta d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos 8\theta}{2} d\theta \\ &= \frac{1}{32} \int_0^{16\pi} \frac{1 - \cos u}{2} du \qquad \text{where } u = 8\theta \text{ and } d\theta = du/8 \\ &= \frac{\pi}{2}. \end{aligned}$$