MATH 162

FINAL EXAM ANSWERS

December 17, 2006

Part A

1. (11 points) Find the volume of the solid obtained by rotating about the y-axis the region under the curve $y = \sqrt{1-x^2}$, for $1/2 \le x \le 1$.

Answer:

Using the shell method, the radius of a shell is x, the height is $y = \sqrt{1-x^2}$, and so the volume of the shell is $2\pi x\sqrt{1-x^2}$. Integrating for $1/2 \le x \le 1$,

$$V = \int_{1/2}^{1} 2\pi x\sqrt{1-x^2}\,dx.$$

Let $u = 1 - x^2$ and then $du = -2x\,dx$. When $x = 1/2$, $u = 3/4$ and when $x = 1$, $u = 0$. Therefore,

$$\begin{aligned} V &= \pi\int_0^{3/4} \sqrt{u}\,du \\ &= \frac{2}{3}\pi u^{3/2}\Big|_0^{3/4} \\ &= \frac{\pi\sqrt{3}}{4}. \end{aligned}$$

The method of cylinders can also be used. We slice the solid perpendicular to the y-axis and integrate with respect to y. A cross-section is an annulus with outer radius $x = \sqrt{1-y^2}$ and inner radius $1/2$. Then the volume is given by the integral

$$\begin{aligned} V &= \pi\int_0^{\sqrt{3}/2} \left[(1-y^2) - \frac{1}{4}\right] dy \\ &= \pi\left[\frac{3}{4}y - \frac{y^3}{3}\right]\Big|_0^{\sqrt{3}/2} \\ &= \frac{\pi\sqrt{3}}{4}. \end{aligned}$$

2. (11 points) A large bathtub has the shape of a hemisphere (half of a sphere) of radius 5 feet, with the center at ground level. It is full of water from the bottom to ground level. How much work is done in pumping the water to the top? Remember that the weight of water is 62.4 pounds per cubic foot.

Answer:

Let x be the distance below ground level, in feet. A slice of water which is at level x, with thickness Δx, would have a circular shape with radius $r = \sqrt{5^2 - x^2}$. Therefore, its volume would be $\pi r^2 \Delta x = \pi(25 - x^2)\Delta x$. The weight of the water in that slice would be $62.4\pi r^2 \Delta x = 62.4\pi(25 - x^2)\Delta x$. Since weight is given in pounds in the British system, and pounds is a measure of force, this is also the force on the slice of water. The work done to raise that slice to the top of the tank would be x times the force, or $62.4\pi x(25 - x^2)\Delta x$. So, the total work done to empty the tank would be

$$\begin{aligned}
\int_0^5 62.4\pi x \left(25 - x^2\right) dx &= 62.4\pi \int_0^5 \left(25x - x^3\right) dx \\
&= 62.4\pi \left(\frac{25x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^5 \\
&= 62.4\pi \left(\frac{25 \times 25}{2} - \frac{625}{4}\right) \\
&= 62.4\pi \cdot \frac{625}{4}
\end{aligned}$$

The work is measured in foot-pounds.

3. (11 points) Solve this indefinite integral:

$$\int x^{3/2} \ln x \, dx$$

Answer:

We use integration by parts with

$$\begin{aligned}
u &= \ln x & dv &= x^{3/2}\, dx \\
du &= \frac{1}{x}\, dx & v &= \frac{2}{5} x^{5/2}
\end{aligned}$$

and we get

$$\begin{aligned}
\int x^{3/2} \ln x \, dx &= \frac{2}{5} x^{5/2} \ln x - \frac{2}{5} \int x^{3/2}\, dx \\
&= \frac{2}{5} x^{5/2} \ln x - \left(\frac{2}{5}\right)^2 x^{5/2} + C.
\end{aligned}$$

4. (11 points) Solve this integral:

$$\int \frac{1}{x^2\sqrt{1-9x^2}}\,dx$$

Answer:

We use the substitution $x = (1/3)\cos(u)$, so that $dx = -(1/3)\sin(u)du$. Then

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{1-9x^2}}\,dx &= -\int = \frac{1}{(1/9)\cos^2(u)\sqrt{1-\cos^2(u)}}\,(1/3)\sin(u)\,du \\
&= -3\int \frac{1}{\cos^2(u)}\,du \\
&= -3\int \sec^2(u)\,du \\
&= -3\tan(u) + C
\end{aligned}$$

Drawing a triangle, we see that $-3\tan(u) + C$ reduces to

$$-\frac{\sqrt{1-9x^2}}{x} + C$$

The problem can also be solved using the substitution $x = (1/3)\sin(u)$, since the integral of $\csc^2(u)$ is $-\cot(u) + C$.

5. (11 points) Solve this integral:

$$\int \frac{1}{t^2(1-t)}\,dt$$

Answer:

We use partial fractions:

$$\frac{1}{t^2(1-t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1-t}$$

Then bringing to a common denominator,

$$\begin{aligned}
1 &= At(1-t) + B(1-t) + Ct^2 \\
1 &= (C-A)t^2 + (A-B)t + B,
\end{aligned}$$

and it follows that $A = B = C = 1$. So the integral becomes

$$\begin{aligned}
\int \frac{1}{t^2(1-t)}\,dt &= \int \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1-t}\right)dt \\
&= \ln|t| - \frac{1}{t} - \ln|1-t| + C'
\end{aligned}$$

6. (12 points) Find

$$\lim_{n\to\infty} \frac{(2n)!}{n^n}$$

You must justify your answer.

Answer:

We might expect the answer to be 0, since $n!$ does not grow as quickly as n^n. However, it might be different because we have $(2n)!$. In fact, we can pair up the factors as follows.

$$\begin{aligned} \frac{(2n)!}{n^n} &= 1 \cdot 2 \cdot 3 \cdots n \cdot \frac{(n+1)\cdots(2n)}{n^n} \\ &= n! \cdot \frac{n+1}{n} \cdots \frac{2n}{n} \\ &\geq n! \end{aligned}$$

So, $(2n)!/n^n$ is at least as large as $n!$, which tends to ∞ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{(2n)!}{n^n} = \infty$$

7. (11 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1+2n+3n^2}{2+4n^2+6n^3}$$

You must justify your answer.

Answer:

Keeping only the leading terms, we would get

$$\sum_{n=1}^{\infty} \frac{3n^2}{6n^3} = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty$$

We still need to justify our reasoning. It seems logical to use the limit comparison test for the divergent series with terms $b_n = 1/(2n)$. Let a_n be the terms of the original series. Then, factoring out the leading terms in a_n and taking the ratio with b_n, we get we get

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{1+2n+3n^2}{2+4n^2+6n^3} \bigg/ \frac{1}{2n} \\ &= \frac{1/n^2+2/n+3}{2/n^3+4/n+6} \cdot \frac{n^2}{n^3} \cdot 2n \\ &\to \frac{3}{6} \cdot 2 \\ &= 1 \end{aligned}$$

Since a_n/b_n tends to a limit which is not 0 or ∞, the limit comparison tests says that $\sum a_n$ and $\sum b_n$ must have the same convergence properties. Since $\sum b_n$ diverges, it follows that $\sum a_n$ diverges.

Thus, the original series diverges.

8. (11 points) Is the following series absolutely convergent, conditionally convergent or divergent?

$$\sum_{n=5}^{\infty}(-1)^n \frac{1}{n \ln n}$$

You must justify your answer.

Answer:

First we note this is an alternating series. We use the Alternating Series Test to see if the series is convergent. It is clear that

- $\displaystyle\lim_{n\to\infty} \frac{1}{n \ln n} = 0,$
- $\displaystyle\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$

Hence the series converges.

Now we want to check if it is absolutely convergent, that is, if

$$\sum_{n=5}^{\infty} \frac{1}{n \ln n}$$

converges. We use the Integral Test:

$$\begin{aligned}
\int_5^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{t\to\infty} \int_5^{t} \frac{1}{x \ln x}\, dx \\
&= \lim_{t\to\infty} \ln(\ln x)\Big|_5^t \\
&= \infty.
\end{aligned}$$

Since the integral diverges, so does $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n \ln n}$ and the original series is NOT absolutely convergent.

Thus, the original series is conditionally convergent.

9. (11 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\cos(2n)}{2^n}$$

You must justify your answer.

Answer:

We can use the Root Test:

$$\lim_{n\to\infty} \left| \frac{\cos(2n)}{2^n} \right|^{1/n} = \lim_{n\to\infty} \frac{|\cos(2n)|^{1/n}}{2} \leq \lim_{n\to\infty} \frac{1^{1/n}}{2} = \frac{1}{2}.$$

Since the Root Test give a number less than one, the series is absolutely convergent.

We can also use the Comparison Test to show the series $\sum_{n=1}^{\infty} \frac{|\cos(2n)|}{2^n}$ is convergent. Since $|\cos(2n)| \leq 1$,

$$\sum_{n=1}^{\infty} \frac{|\cos(2n)|}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n},$$

and the series on the right hand side is a geometric series with $r = 1/2$, hence convergent. We conclude the original series is absolutely convergent.

Part B

10. (13 points) Find a power series representation for

$$\frac{1}{2+x^2}$$

You must show your reasoning. Secondly, find the radius of convergence.

Answer:

We recall that the geometric series has the following sum.

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots$$

valid for $|r| < 1$. Substituting $r = -x^2/2$, we get

$$\frac{1}{1+x^2/2} = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{8} + \cdots$$

valid for $|-x^2/2| < 1$, or in other words, valid for $|x| \leq \sqrt{2}$. Now we just have to divide the equation by 2, which doesn't change the radius, so

$$\frac{1}{2+x^2} = \frac{1}{2} - \frac{x^2}{4} + \frac{x^4}{8} - \frac{x^6}{16} + \cdots$$

also valid for $|x| \leq \sqrt{2}$.

11. (13 points) Let $f(x) = (x-3)^{10}$.

(a) Find the Taylor series for $f(x)$, centered at $a = 3$.

(b) Find $f^{(5)}(3)$

(c) Find $f^{(10)}(3)$

Answer:

(a) $f(x)$ is already in the form of its Taylor series, centered at $a = 3$. All terms except $(x-3)^{10}$ have coefficient zero. That is, if we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-3)^n,$$

then $c_{10} = 1$ and $c_n = 0$ for all $n \neq 10$.

(b) Recall that $c_5 = \frac{f^{(5)}(3)}{5!}$ and since $c_5 = 0$, we have $f^{(5)}(3) = 0$

(c) Similarly to (b), $f^{(10)}(3) = 10! c_{10} = 10!$

12. (13 points) Suppose we approximate e^x with its Taylor series up to x^3.

$$T_3(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Find a number $A > 0$ such that for $x \in [-A, 0]$,

$$|e^x - T_3(x)| \leq 10^{-4}$$

Answer:

The error estimate in Taylor's formula states that for $x \in [-A, 0]$,

$$|e^x - T_3(x)| \leq \frac{M}{4!}|x|^4$$

where M is the largest value of $|(e^x)''''| = e^x$ for $x \in [-A, 0]$. However, if $x \in [-A, 0]$ then $|x|^4 \leq |A|^4$ and the largest value of e^x occurs at $x = 0$, which gives $M = 1$. Therefore, we must find A such that

$$\frac{1}{4!} A^4 = 10^{-4}$$

This gives

$$A = \frac{(4!)^{1/4}}{10} = \frac{24^{1/4}}{10}$$

13. (12 points) Find the length of the curve $y = \frac{\ln x}{2} - \frac{x^2}{4}$, for $1 \leq x \leq 2$.

Answer:

First, $\frac{dy}{dx} = \frac{1}{2x} - \frac{x}{2}$, and the length of the curve is then given by

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + \left(\frac{1}{4x^2} - \frac{1}{2} + \frac{x^2}{4}\right)}\, dx \\ &= \int_1^2 \sqrt{\frac{1}{4x^2} + \frac{1}{2} + \frac{x^2}{4}}\, dx \\ &= \int_1^2 \sqrt{\left(\frac{1}{2x} + \frac{x}{2}\right)^2}\, dx \\ &= \int_1^2 \left(\frac{1}{2x} + \frac{x}{2}\right) dx \\ &= \left(\frac{\ln x}{2} + \frac{x^2}{4}\right)\Big|_1^2 \\ &= \frac{\ln 2}{2} + \frac{3}{4} \end{aligned}$$

14. (12 points) Find the area of the surface generated by rotating the curve

$$y = x^3$$

about the x-axis, where x lies between 1 and 3.

Answer:

Using the formula for the area of a surface rotated about the x-axis, we find

$$\begin{aligned} A &= \int_1^3 2\pi y ds \\ &= \int_1^3 2\pi y \sqrt{dx^2 + dy^2} \\ &= \int_1^3 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}$$

Since $y = x^3$ and $dy/dx = 3x^2$,

$$\begin{aligned} A &= 2\pi \int_1^3 x^3 \sqrt{1 + (3x^2)^2}\, dx \\ &= 2\pi \int_1^3 x^3 \sqrt{1 + 9x^4}\, dx \end{aligned}$$

Using the substitution $u = 1 + 9x^4$, $du = 36x^3 dx$ we get

$$\begin{aligned} A &= \frac{2\pi}{36} \int_{10}^{3^6+1} u^{1/2}\, du \\ &= \frac{2\pi}{36} \cdot \frac{2}{3} u^{3/2} \Big|_{10}^{3^6+1} \\ &= \frac{\pi}{27} \left([3^6 + 1]^{2/3} - 10^{3/2} \right) \end{aligned}$$

15. (12 points) Suppose that a parametric curve is defined by the following equation.

$$\begin{aligned} x(t) &= te^t \\ y(t) &= \sin(t) \end{aligned}$$

Write an integral which represents the length of the curve between $t = 1$ and $t = 2$.

DO NOT SOLVE THE INTEGRAL.

Your integral must be written in terms of the functions $t, e^t, \ln(t)$ and trig functions. It should not explicitly involve $x(t)$ or $y(t)$.

Answer:

We have

$$\begin{aligned} L &= \int_1^2 ds \\ &= \int_1^2 \sqrt{dx^2 + dy^2} \\ &= \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_1^2 \sqrt{(e^t + te^t)^2 + \cos^2 t}\, dt \end{aligned}$$

16. (13 points) Find the points on the polar curve $r \sin\theta = 1$ where the tangent line is horizontal.

Answer:

Recall that $y = r\sin\theta$, and thus $y = 1$ which implies $\dfrac{dy}{dx} = 0$. In other words, the tangent to this curve is always horizontal.

If we did not observe $y = 1$, using the slope formula for polar curves, and $r = \dfrac{1}{\sin\theta}$, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{-\cos\theta}{\sin^2\theta}\sin\theta + \frac{1}{\sin\theta}\cos\theta}{\frac{-\cos\theta}{\sin^2\theta}\cos\theta - \frac{1}{\sin\theta}\sin\theta} \\
&= 0
\end{aligned}$$

17. (12 points) Consider the polar curve

$$r = 3\theta^3, \qquad 0 \le \theta \le 2\pi$$

Find the area of the region bounded by the curve and the ray $\theta = 0$, $r \ge 0$.

Answer:

Using the area formula for polar curves, we get

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} \left(3\theta^3\right)^2 \, d\theta \\
&= \frac{9}{2}\int_0^{2\pi} \theta^6 \, d\theta \\
&= \frac{9}{2} \cdot \frac{\theta^7}{7}\Big|_0^{2\pi} \\
&= \frac{9}{14}(2\pi)^7
\end{aligned}$$