MATH 162

Midterm 2

April 8th, 2025

Name: SOLUTIONS

UR ID: ______________________

Circle your Instructor's Name:

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Instructions:

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• For each problem, the points are equally distributed among parts.

• When applicable, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$.

• This exam is out of 100 points. You are responsible for checking that this exam has all 15 pages.

• You have 75 minutes for this exam.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______________________

QUESTION	VALUE	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

Trig formulas

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta \qquad \sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

Integration by parts formulas:

- $\int u dv = uv - \int v du$
- $\int \tan x dx = \ln|\sec x| + C \qquad \int \sec x dx = \ln|\sec x + \tan x| + C$

Area of surface of revolution from $y = f(x)$, $a \leq x \leq b$:

- Rotation about x-axis: $S = 2\pi \int_a^b f(x)\sqrt{(f'(x))^2+1}\ dx$
- Rotation about y-axis: $S = 2\pi \int_a^b x\sqrt{(f'(x))^2+1}\ dx$

Some formulas for parametric equations:

- $dy/dx = \frac{dy/dt}{dx/dt} \qquad ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$

Polar coordinates and polar curves:

- $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$. Area bounded by the polar curve: $S = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$

1. (20 points) Answer the following questions about improper integrals, justifying your answer.

(a) Use comparison theorem to determine whether the integral is convergent or divergent.

$$\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+1}}\,dx.$$

$$\left.\begin{array}{l} 0 \leq \dfrac{2+\cos x}{\sqrt{x^4+1}} \leq \dfrac{3}{\sqrt{x^4+1}} < \dfrac{3}{x^2} \\ \text{By p-test } \displaystyle\int_1^\infty \frac{3}{x^2}\,dx \text{ converges} \end{array}\right\} \to \int_1^\infty \frac{2+\cos x}{\sqrt{x^4+1}}\,dx \text{ converges.}$$

Answer:

(b) Determine whether the following integral is convergent or divergent. If it is convergent, evaluate the value of the integral.

$$\int_2^\infty \frac{dx}{x(\ln x)^3}.$$

$$\left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right.$$

$$\int_{\ln 2}^{\infty} \frac{1}{u^3}\,du = \lim_{t\to\infty} \int_{\ln(2)}^{t} \frac{1}{u^3}\,du = \lim_{t\to\infty} \left[\frac{-1}{2u^2}\right]_{\ln(2)}^{t}$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2t^2} + \frac{1}{2(\ln(2))^2}\right] = \frac{1}{2[\ln 2]^2}$$

Answer:

2. (20 points) Answer the following, justifying your answer.

(a) Find the exact length of the curve $y = \ln(\sec(x))$, $\quad 0 \leq x \leq \pi/4$.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{\sec x \tan x}{\sec x}\right)^2}\, dx$$

$$= \sqrt{1 + \tan^2 x}\, dx = \sec x\, dx$$

$$\Rightarrow \text{Arclength} = \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1).$$

Answer:

(b) Find the exact area of the surface obtained by rotating the curve $y = \sqrt{1+e^x}$, $\quad 0 \le x \le 1$ about the x-axis.

$$\text{Area} = 2\pi \int_{x=0}^{x=1} y \, ds = 2\pi \int_0^1 \sqrt{1+e^x} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_0^1 \sqrt{1+e^x} \sqrt{1+\left(\frac{e^x}{2\sqrt{e^x+1}}\right)^2} \, dx$$

$$= 2\pi \int_0^1 \sqrt{1+e^x} \sqrt{\frac{4e^x+4+e^{2x}}{4(e^x+1)}} \, dx$$

$$= 2\pi \int_0^1 \frac{1}{2}(e^x+2) \, dx = \pi \left[e^x + 2x\right]_0^1$$

$$= \pi(e+2) - \pi(1) = \pi(e+1)$$

Answer:

3. (20 points) Answer the following, justifying your answer.

(a) Compute the indefinite integral $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$.

$$= \int \frac{1}{\sqrt{(x+1)^2+4}}\, dx \xrightarrow{\text{trig sub}} \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$$\begin{cases} x+1 = 2\tan\theta \\ dx = 2\sec^2\theta\, d\theta \\ (x+1)^2+4 = 4\sec^2\theta \end{cases}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{1}{2}\sqrt{(x+1)^2+4} + \frac{(x+1)}{2}\right| + C$$

Answer:

(b) Write down the partial fraction decomposition of the following function, without finding the constants involved in the decomposition:

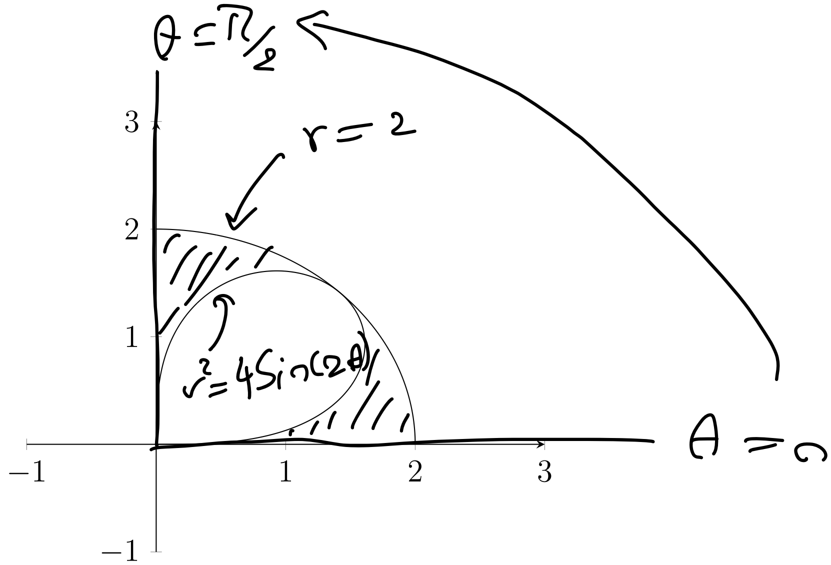
$$f(x) = \frac{6x^2 - 2x + 16}{x^4 + 4x^2} = \frac{P(x)}{Q(x)}$$

$\rightarrow$ $Q(x) = x^2(x^2+1)$

$\rightarrow$ PFD: $f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

Answer:

4. (20 points) Consider the region **in the first quadrant** which lies inside the circle $r = 2$ and outside the *lemniscate* $r^2 = 4\sin(2\theta)$.



(a) Write a definite integral whose value measures the area of such region.

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} \left(\underset{\text{polar}}{\text{outer}}\right)^2 - \left(\underset{\text{polar}}{\text{inner}}\right)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(4 - 4\sin(2\theta)\right) d\theta$$

Answer:

(b) Compute the area of the described region by evaluating such integral.

$$\text{Area} = 2\left[\theta + \frac{\cos(2\theta)}{2}\right]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - \frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = \pi - 2$$

Answer:

5. (20 points) Answer the following, justifying your answer.

(a) Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{\left(n^4 \sin^2\left(\frac{1}{n^2}\right)\right)\right\}_{n=1}^{\infty}.$$

$\lim \left(n^4 \sin^2 \frac{1}{n^2}\right) = \lim\limits_{x \to \infty} x^4 \sin^2 \frac{1}{x^2} \overset{y = 1/x^2}{=\!=\!=} \lim\limits_{y \to 0^+} \frac{\sin^2 y}{y^2}$

$\to \left(\lim\limits_{y \to 0} \frac{\sin y}{y}\right)^2 = 1$

Answer:

(b) Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}.$$

$= \{\overbrace{1, 0, -1, 0,}^{\text{repeat}}\ 1, 0, -1, 0, \cdots\}$

$\to$ sequence "oscillates"

& therefore diverges.

Answer:

(c) Find all the horizontal tangencies of the parametric curve

$$\begin{cases} x = t^5 + t^3 \\ y = t^2 + 5 \end{cases},$$

or show that it does not have any.

Slope of the tangent $= \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$= \dfrac{2t}{5t^4+3t^2} = \dfrac{2}{5t^3+3t} \neq 0$ for any $t \in \mathbb{R}$

$\rightarrow$ no horizontal tangency.

Answer:

EXTRA PAGE. You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.

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