Math 162: Calculus IIA

Midterm 2 ANSWERS

March 16, 2022

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Nontrivial integrals of trig functions.

$$\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec(x) + \tan(x)| + C$$

1. (20 points)

(a) Evaluate the integral $\int_0^{\pi/4} \sec^4(x)\tan^3(x)\,dx$.

Answer:

$$\begin{aligned}
\int_0^{\pi/4} \sec^4(x)\tan^3(x)\,dx &= \int_0^{\pi/4} \sec^2(x)\tan^3(x)\sec^2(x)\,dx \\
&= \int_0^{\pi/4} (1+\tan^2(x))\tan^3(x)\sec^2(x)\,dx \\
&\text{let } u = \tan(x), du = \sec^2(x)\,dx \\
&= \int_0^1 (1+u^2)u^3\,du \\
&= \int_0^1 u^3 + u^5\,du \\
&= \left(\frac{u^4}{4} + \frac{u^6}{6}\right)\Big|_0^1 \\
&= \frac{1}{4} + \frac{1}{6}
\end{aligned}$$

(b) Evaluate the indefinite integral $\int \frac{1}{\sqrt{x^2+2x+2}}\,dx$.

Answer:

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+2x+2}}\,dx &= \int \frac{1}{\sqrt{(x+1)^2+1}}\,dx \\
&\text{let } x+1 = \tan(\theta) \text{ so } dx = \sec^2(\theta)\,d\theta \\
&= \int \frac{1}{\sqrt{\tan^2(\theta)+1}}\sec^2(\theta)\,d\theta \\
&= \int \frac{1}{\sec(\theta)}\sec^2(\theta)\,d\theta \\
&= \int \sec(\theta)\,d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C
\end{aligned}$$

If $x+1 = \tan(\theta)$, then $\sec(\theta) = \sqrt{x^2+2x+2}$, so

$$\int \frac{1}{\sqrt{x^2+2x+2}}\,dx = \ln|\sqrt{x^2+2x+2} + (x+1)| + C$$

2. (20 points)

Evaluate

$$\int \frac{x^3+2x^2-x+2}{x^2\left(x^2+1\right)}\,dx$$

Hint: Use the method of partial fractions.

Answer:

x^2+1 is an irreducible quadratic since x^2 can never be -1 and x^2 is a quadratic which has the repeated linear factor x, so by partial fractions method:

$$\frac{x^3+2x^2-x+2}{x^2\left(x^2+1\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$$

where A, B, C, and D are constants. So,

$$x^3+2x^2-x+2=Ax\left(x^2+1\right)+B\left(x^2+1\right)+Cx^3+Dx^2$$

Expanding the right hand side we obtain

$$x^3+2x^2-x+2=Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$$

This is an equation of polynomials so all coefficients should be set equal to each other:

$$\begin{aligned}1x^3&=(A+C)\,x^3\\2x^2&=(B+D)\,x^2\\-x&=Ax\\2&=B\end{aligned}$$

So $B=2$ and $A=-1$. By the first two equations we see that $A+C=1$ and $B+D=2$. Therefore, $C=2$ and $D=0$. Therefore,

$$\int \frac{x^3+2x^2-x+2}{x^2\left(x^2+1\right)}\,dx=\int \frac{-1}{x}\,dx+\int \frac{2}{x^2}\,dx+\int \frac{2x}{x^2+1}\,dx$$

For the third integral, let $u=x^2+1$, then $du=2x\;dx$

$$\int \frac{2x}{x^2+1}\,dx=\int \frac{du}{u}=\ln|u|=\ln\left(x^2+1\right)$$

up to addition by a constant.

Therefore,

$$\int \frac{x^3 + 2x^2 - x + 2}{x^2\left(x^2+1\right)} = -\ln|x| - \frac{2}{x} + \ln\left(x^2+1\right) + C.$$

3. (20 points)

For each of the following improper integrals, explain why it is improper, and determine whether it converges or diverges. If it converges, then show to what it converges. Be sure to SHOW ALL WORK.

(a) $\displaystyle\int_0^9 (x-1)^{-1/3}dx$

Answer:

The integrand is not defined for $x = 1$, making the integral improper. The improper integral will converge if and only if each of the improper integrals

$$\int_0^1 (x-1)^{-1/3}dx$$

and

$$\int_1^9 (x-1)^{-1/3}dx$$

both converge. An antiderivative of $(x-1)^{-1/3}$ is $(3/2)(x-1)^{2/3}$, so

$$\int_0^1 (x-1)^{-1/3}dx = \lim_{t\to 1-}\int_0^t (x-1)^{-1/3}dx = \lim_{t\to 1-}(3/2)[(t-1)^{2/3} - (-1)^{2/3}] = -3/2$$

$$\int_1^9 (x-1)^{-1/3}dx = \lim_{t\to 1+}\int_t^9 (x-1)^{-1/3}dx = \lim_{t\to 1+}(3/2)[-(t-1)^{2/3}+(8)^{2/3}] = 3/2[0+4] = 6$$

Hence the improper integral converges to $6 + (-3/2) = 9/2$.

(b) $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

Answer:

The integrand is not defined for $x = 0$; therefore the integral is improper.

First, let us find an antiderivative of $e^{1/x}/x^3$. Make the substitution $u = 1/x$, so $du = -1/x^2\,dx$, so

$$\int \frac{e^{1/x}}{x^3}\,dx = \int e^u u^3 \cdot -u^{-2}du = -\int ue^u du$$

The last integral is easily evaluated by parts: $U = u$, $dV = e^u du$, so $dU = du$, $V = e^u$; $\int U dV = UV - \int V dU$; so

$$\int ue^u du = ue^u - \int e^u du = ue^u - e^u = e^u(u-1)$$

So

$$\int \frac{e^{1/x}dx}{x^3} = -e^{1/x}\left(\frac{1}{x} - 1\right) + C = e^{1/x}\left(1 - \frac{1}{x}\right) + C$$

Thus

$$\int_{-1}^{0} \frac{e^{1/x}dx}{x^3} = \lim_{t\to 0-} \int_{-1}^{t} \frac{e^{1/x}dx}{x^3} = \lim_{t\to 0-} e^{1/t}\left(1 - \frac{1}{t}\right) - e^{-1}\left(1 - \frac{1}{-1}\right) = -2/e + \lim_{t\to 0-} \frac{1-(1/t)}{e^{-1/t}}$$

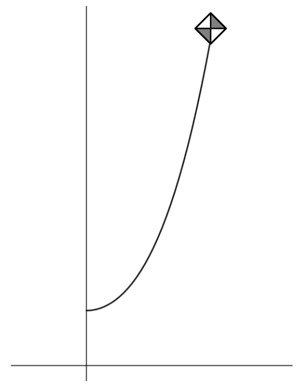
The last limit obeys the hypotheses for l'Hopital's Rule, so

$$\int_{-1}^{0} \frac{e^{1/x}dx}{x^3} = -2/e + \lim_{t\to 0-} \frac{1/t^2}{e^{-1/t}\cdot(1/t^2)} = -2/e + \lim_{t\to 0-} \frac{1}{e^{-1/t}} = -2/e$$

4. (10 points)

The kite's height (in m) above ground with respect to its horizontal position is given by the function

$$y = \frac{1}{6}(x^2 + 4)^{3/2}$$



A steady wind blows a kite due west. Find the distance traveled by the kite from $x = 0$ to $x = 3$.

Answer:

Apply the arc length formula

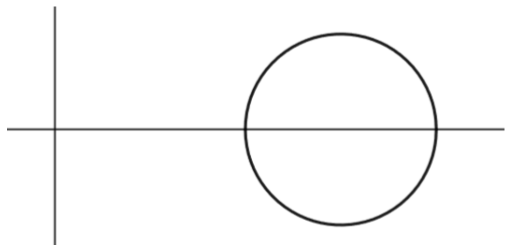
$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Since $f'(x) = \frac{1}{2}x(x^2 + 4)^{1/2}$, it holds that $(f'(x))^2 = \frac{1}{4}x^2(x^2 + 4) = \frac{x^4}{4} + x^2$. Therefore,

$$L = \int_0^3 \sqrt{1 + \frac{x^4}{4} + x^2}\, dx = \int_0^3 \sqrt{\left(\frac{x^2}{2} + 1\right)^2}\, dx$$
$$= \int_0^3 \frac{x^2}{2} + 1\, dx = \frac{x^3}{6} + x \,\Big|_0^3 = \frac{9}{2} + 3 = \frac{15}{2} m.$$

5. (10 points)

(a) The local bakery makes chocolate dip donuts. The surface of the donut can be described as the surface of revolution obtained by revolving the circumference $(x-3)^2+y^2=1$ about the y-axis. A sketch of the curve is shown below



Assuming that only the upper half of the donut is evenly dipped, write an integral (**do not evaluate it!**) that represents the surface area of the donut covered by chocolate.

Answer:

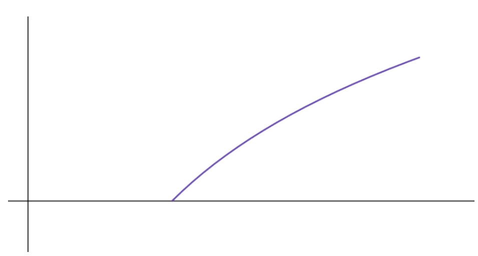
Apply the surface area formula (revolving about the y-axis)

$$S = 2\pi \int_a^b x\sqrt{1+(f'(x))^2}dx.$$

The upper half of the donut is given by the function $f(x) = \sqrt{1-(x-3)^2}$ and so $f'(x) = -\frac{x-3}{\sqrt{1-(x-3)^2}}$. The surface area is represented by the integral

$$S = 2\pi \int_2^4 x\,\sqrt{1+\frac{(x-3)^2}{1-(x-3)^2}}\,dx$$

(b) The bakery adds a chocolate dipped strawberry on top of the donut. The strawberry can be described as the surface of revolution obtained by revolving the curve $y=\ln(x)$ about the x-axis from $x=1$ to $x=\sqrt{e}$. A sketch of the curve is shown below.



Write an integral (**do not evaluate it!**) that represents the surface area of the strawberry covered by chocolate.

Answer:

Apply the surface area formula (revolving about the x-axis)

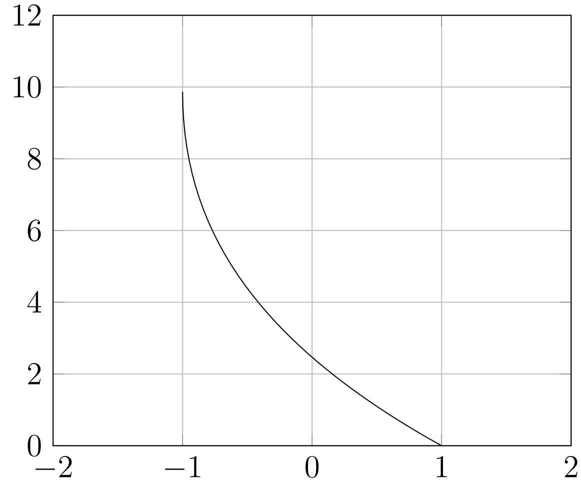
$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

Since the surface of the strawberry is given by $f(x) = \ln(x)$, the surface area is

$$S = 2\pi \int_1^{\sqrt{e}} \ln(x)\ \sqrt{1 + \frac{1}{x^2}}\ dx$$

6. (20 points)

Consider the parametric curve defined by $x(t) = \cos(t)$ and $y(t) = t^2$ for $0 \leq t \leq \pi$.



(a) For $0 \leq t \leq \pi$, find the t values where the tangent line has vertical slope and the t values where the tangent line has horizontal slope.

Answer:

The slope of the tangent line is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-\sin(t)}$. It is horizontal when $\frac{dy}{dx} = 0$ and vertical when $\frac{dy}{dx} = \infty$.

When $t = 0$, both the numerator and denominator are 0, so we take a limit $\lim_{t\to 0} \frac{2t}{-\sin(t)}$ and get that the slope at $t = 0$ is -2. The numerator is otherwise never 0, so there is no horizontal slope.

The denominator is also 0 when $t = \pi$, so there is a vertical slope for $t = \pi$.

(b) Write down **but do not evalute** an integral (with respect to t) expressing the area under the curve from $0 \leq t \leq \pi$.

Answer:

Observe that since $x(t) = \cos(t)$ is a decreasing function from 0 to π, we integrate from π to 0 to find the area under the curve.

We use the formula:

$$\int y\,dx = \int_{t=\pi}^{t=0} y(t)\,\frac{dx}{dt}dt = \int_{t=\pi}^{t=0} t^2(-\sin(t))\,dt.$$

Scratch paper

More scratch paper