Math 162: Calculus IIA

Midterm 1 ANSWERS

February 21, 2022

1. (20 points)

(a) Evaluate the integral $\int_0^{\pi/4} \frac{4x-6}{1+x^2}\,dx$.

Answer:

$$\begin{aligned} I = \int_0^{\pi/4} \frac{4x-6}{1+x^2}\,dx &= \int_0^{\pi/4} \frac{4x}{1+x^2}\,dx - \int_0^{\pi/4} \frac{6}{1+x^2}\,dx \\ &= \int_0^{\pi/4} \frac{4x}{1+x^2}\,dx - 6\int_0^{\pi/4} \frac{1}{1+x^2}\,dx \end{aligned}$$

The second integral is

$$\arctan(x)\Big|_0^{\frac{\pi}{4}} = \arctan\left(\frac{\pi}{4}\right) - \arctan(0) = \arctan\left(\frac{\pi}{4}\right)$$

For the first integral, we do the following u-substitution:

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2x\;dx \end{aligned}$$

The lower u-bound is $1 + 0^2 = 1$, the upper u-bound is $1 + \frac{\pi^2}{16}$. Hence our expression becomes:

$$\begin{aligned} &= -6\arctan\left(\frac{\pi}{4}\right) + \int_1^{\left(1+\frac{\pi^2}{16}\right)} \frac{2}{u}\,du \\ &= -6\arctan\left(\frac{\pi}{4}\right) + 2\ln\Big|_1^{1+\frac{\pi^2}{16}} \\ &= -6\arctan\left(\frac{\pi}{4}\right) + 2\left(\ln\left(1+\frac{\pi^2}{16}\right) - \ln(1)\right) \\ &= -6\arctan\left(\frac{\pi}{4}\right) + 2\ln\left(1+\frac{\pi^2}{16}\right). \end{aligned}$$

(b) Evaluate the integral $\int 2x^5\sqrt{x^2-2}\,dx$.

Answer:

Consider the u-substitution $u = x^2 - 2$, then

$$du = 2x\ dx$$

Observe that

$$2x^5\ dx = (2x)\,x^4\ dx = (u+2)^2\ du$$

since $x^4 = (u+2)^2$. Therefore,

$$I = \int 2x^5\sqrt{x^2-2}\ \ dx = \int (u+2)^2\sqrt{u}\ \ du$$

Expanding the square we get

$$\begin{aligned}
&= \int \left(u^2+4u+4\right)\sqrt{u}\ \ du \\
&= \int u^{5/2} + 4u^{3/2} + 4u^{1/2}\ \ du \\
&= \frac{2}{7}u^{7/2} + \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C \\
&= \frac{2}{7}(x^2-2)^{7/2} + \frac{8}{5}(x^2-2)^{5/2} + \frac{8}{3}(x^2-2)^{3/2} + C
\end{aligned}$$

where C is a constant.

2. (20 points)

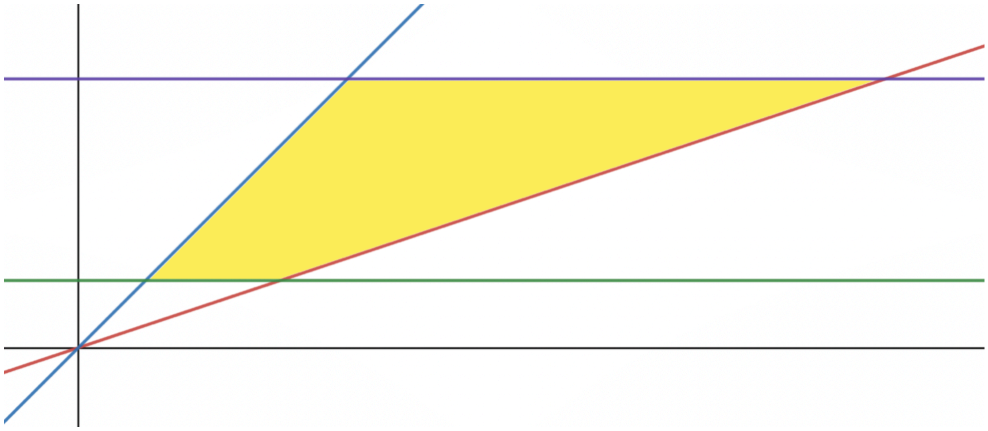
(a) The lines $y = x, y = x/3, y = 1$, and $y = 4$ enclose a region in the plane, as shown below:

What is the area of the region enclosed by the lines?

Answer:

To describe the area of the region using a single integral, we note that we must integrate with respect to y, as otherwise there are multiple pairs of bounding curves for the region. When we convert the bounding curves $y = x$ and $y = x/3$ into $x = y$ and $x = y/3$, we see that the region is precisely the region lying between $x = y$ and $x = 3y$ for $1 \le y \le 4$. Therefore, the area of the region is

$$A = \int_1^4 3y - y\,dy = \int_1^4 2y\,dy = y^2\Big|_1^4 = 15$$



(b) Find the value of c such that the horizontal line $y = c$ divides the region from part (a) into two regions of equal area.

Answer:

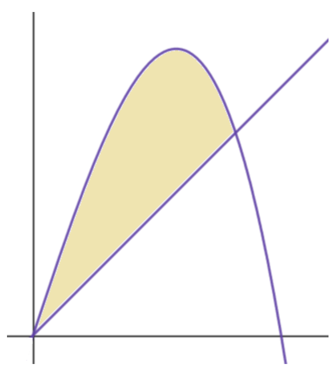
As the area of the region is 15, we wish to find c such that $1 \leq c \leq 4$ and

$$\frac{15}{2} = \int_1^c 3y - y\,dy = \int_1^c 2y\,dy = y^2\Big|_1^c = c^2 - 1.$$

It follows quickly that $c = \sqrt{\frac{15}{2} + 1}$

3. (20 points)

(a) Consider the region in the first quadrant enclosed by the curve $y = 3x - 2x^3$ and the line $y = x$. A sketch of the region is shown in the following graph:



Write down, **but do not evaluate** an integral expressing the volume of the solid obtained by revolving the region about the y-axis.

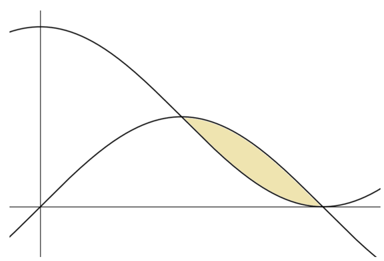
Answer:

We first find the intersection points by solving $3x - 2x^3 = x \Rightarrow 2x - 2x^3 = 0$ so $x = 0, 1$ in the first quadrant. We use the cylindrical shell method to compute the volume of the solid. We have that

$$\begin{aligned} V &= \int_0^1 2\pi x(3x - 2x^3 - x)\,dx \\ &= 4\pi \int_0^1 x(x - x^3)\,dx \\ &= 4\pi \int_0^1 (x^2 - x^4)\,dx \end{aligned}$$

Note: It is not a good idea to use the disk/washer method since it is hard to isolate x from $y = 3x - 2x^3$.

(b) Consider the region enclosed by the curves $y = \sin(x)$ and $y = 1 + \cos(x)$, and the lines $x = \pi/2$ and $x = \pi$. A sketch of the region is shown in the following graph.



Write down, **but do not evaluate** an integral expressing the volume of the solid obtained by revolving the region about the line $y = 1$.

Answer:

We use the disks/washers method to compute the volume of the solid. Since the axis of revolution is the line $y = 1$, the exterior radius of the washer is $R = 1 - (1 + \cos(x)) = -\cos(x)$ and the interior radius of the washer is $r = 1 - \sin(x)$ for x in the interval $[\pi/2, \pi]$. Therefore, we have that

$$\begin{aligned} V &= \pi \int_{\pi/2}^{\pi} (-\cos(x))^2 - (1 - \sin(x))^2\,dx \\ &= \pi \int_{\pi/2}^{\pi} \cos^2(x) - 1 + 2\sin(x) - \sin^2(x)\,dx \\ &= \pi \int_{\pi/2}^{\pi} \cos(2x) - 1 + 2\sin(x)\,dx \end{aligned}$$

4. (20 points)

A pool has a radius of 5 meters and a height of 4 meters. It is filled halfway with an unknown liquid. This liquid is then fully drained from the top of the pool, which takes 1,800,000π Joules of energy. What is the density of the liquid? [Approximate gravity as 10 m/s^2.]

Answer:

The radius of the pool is 5 meters, so the area of a layer of the liquid is 9π squared meters. The volume of a thin layer of water with height dy meters is $25\pi dy$ cubic meters.

Let D be the density. Multiply by the density to get that the mass of this layer is $25D\pi dy$ kilograms. Multiply by gravity to get that the force is $250D\pi dy$ Newtons.

The distance a layer of the liquid has traveled is $4 - y$.

So the total work is

$$\int_0^2 250D\pi(4-y)\,dy = 250D\pi\left[4y - y^2/2\right]\Big|_0^2 = 1500D\pi.$$

Setting $1500D\pi$ J equal to 1,800,000π J and solving for D, we find that $D = 1200\ kg/m^3$.

5. (20 points)

(a) Use Integration by Parts to show that, for every integer $n \geq 1$,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Answer:

Let $u = (\ln x)^n$, $dv = dx$. Then

$$du = \frac{n(\ln x)^{n-1}\,dx}{x} \text{ and } v = x.$$

Now, $\int u\,dv = uv - \int v\,du$, so $\int (\ln x)^n,\ dx = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$.

(b) Use (a) to compute $\displaystyle\int \ln x\,dx$.

Answer:

Taking $n = 1$ in (a) gives $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$

(c) Use (a) and (b) to compute $\int (\ln x)^2\,dx$.

Answer:

Taking $n = 2$ in (a) gives

$$\int (\ln x)^2\,dx = x(\ln x)^2 = x(\ln x)^2 - 2\int \ln x\,dx.$$

Using the formula in (b) this becomes

$$\int (\ln x)^2\,dx = x(\ln x)^2 = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Scratch paper

More scratch paper