Math 162: Calculus IIA

Final Exam ANSWERS

May 2, 2022

HANDY DANDY FORMULAS

Integration by parts formula: $\int u\,dv = uv - \int v\,du$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Area of surface of revolution in rectangular coordinates for $y = f(x)$ with $a \le x \le b$

- about the x-axis: $S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$
- about the y-axis: $S = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx$

Polar coordinates

$$r = \sqrt{x^2+y^2} \qquad \theta = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi + \arctan(y/x) & \text{for } x < 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{for } (x,y) = (0,0) \end{cases}$$

$$x = r\cos\theta, y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

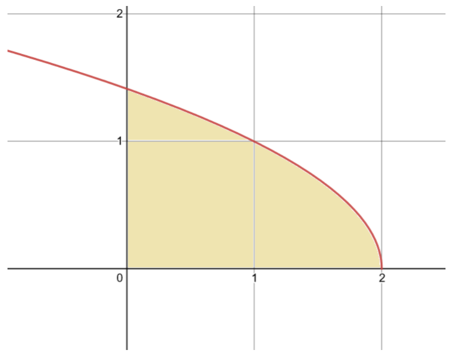
Infinite series formulas

- The *Maclaurin series* for $f(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
- The *Taylor series* for $f(x)$ at a is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.
- The *mth Taylor polynomial* is $T_m(x) = \displaystyle\sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!} (x-a)^n$,
- The *mth Taylor remainder* is $R_m(x) = f(x) - T_m(x)$
- Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then $|R_m(x)| \leq \dfrac{(x-a)^{n+1}M}{(n+1)!}$.

Part A

1. (20 points)

Consider the region in the first quadrant enclosed by the curve $y = \sqrt{2-x}$ and the lines $x = 0$ and $y = 0$. A sketch of the region is shown in the following graph:



(a) (10 points)

Use the **shell method** to write an integral that represents the volume of the solid obtained by revolving the region about the y-axis. **Do not evaluate the integral!**.

Answer:

Using the shell method, we have shells of radius x. height $\sqrt{2-x}$ and thickness Δx. Therefore,

$$V = 2\pi \int_0^2 x\sqrt{2-x}\, dx.$$

(b) (10 points) Use the **disk/washer method** to write an integral that represents the volume of the solid obtained by revolving the region about the line $y = 2$. **Do not evaluate the integral!**.

Answer:

Using the washer method, we have washers with exterior radius 2, interior radius $2 - \sqrt{2-x}$ and thickness Δx. Therefore,

$$V = \pi \int_0^2 (2^2 - (2 - \sqrt{2-x})^2)\, dx.$$

2. (20 points) A rectangular water tank that is 5 meters long, 3 meters wide, and 12 meters deep is full of water. Take the density of water as 1000 kg/m^3 and approximate the acceleration due to gravity as 10 m/s^2.

(a) (10 points) Set up **but do not evaluate** the integral expression for the work needed to pump all of the water out of the tank.

Answer:

The area of a layer of water is $3 \cdot 5 = 15$ squared meters as it is 3 meters wide and 5 meters long.
The volume of a thin layer of water with height dy meters is 15 dy cubic meters.

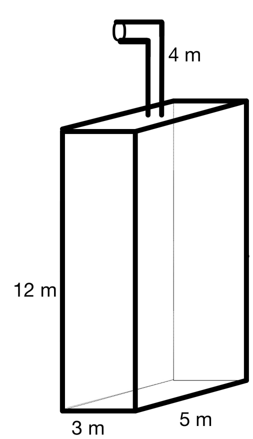
Multiplying this by the density of the water, $1000kg/m^3$, we obtain the mass of the layer as $15,000\ dy$ kilograms. Multiplying this by gravity we see that force is $150,000\ dy$ Newtons.

The distance a layer of water travels is $12 - y$ meters because the measured distance is between the height of a layer, y meters, and the top of the tank, 12 meters above the ground.

As we pump all the water out the lower bound of integration is $y = 0$ and as it is full of water the upper bound is $y = 12$. Hence the total work is expressed by

$$\int_0^{12} 150,000(12 - y)\ dy$$

(b) (10 points) Set up **but do not evaluate** the integral expression for the work needed to pump one fourth of the water out with a pipe that is 4 m above the top of the tank.



Answer:

For part (b), the force computation is exactly the same as part (a), so the force is $150,000\ dy$ Newtons because the set up in part (b) only affects the distance that a thin water layer takes and the integral bounds.

Here, a layer travels up to the top of the tank but then goes through the pipe and hence it is $(12 - y) + 4 = 16 - y$ meters.

As we pump one fourth of the water out, we stop, when the height of the water is three fourth of the depth of the tank, hence $y = 9$ is the lower bound. As the tank is full, the upper bound will stay the same as $y = 12$. Hence the total work is expressed by

$$\int_9^{12} 150,000(16 - y)\ dy$$

3. (15 points) Compute the following integral:

$$\int 6 \arctan\left(\frac{8}{w}\right)\ dw$$

Answer:

Take $u = 6 \arctan\left(\frac{8}{w}\right)$, $dv = dw$. So $du = \frac{-48}{w^2+64}\ dw$, $v = w$. Then,

$$\int 6 \tan^{-1} \frac{8}{w}\ dw = 6w \tan^{-1} \frac{8}{w} + 48 \int \frac{w}{w^2 + 64}\ dw = 6w \tan^{-1} 8w + 24 \ln |w^2 + 64| + C.$$

4. (15 points)

Compute the following integral:

$$\int \frac{\sqrt{4 - x^2}}{x^4}\ dx$$

Answer:

Let $\cos(\theta) = x/2$. So $dx = -\sin(\theta)/2d\theta$. $\sqrt{4 - x^2} = \sin(\theta)$.

After trig substitution, the integral is

$$-\int \frac{\sin^2(\theta)}{32 \cos^4(\theta)} d\theta.$$

Rewritten, this is

$$-\frac{1}{32}\int \tan^2(\theta)\sec^2(\theta)d\theta.$$

Set $u = \tan(\theta)$, so $du = \sec^2(\theta)d\theta$. After this substitution the integral is

$$-\frac{1}{32}\int u^2 du.$$

This equals

$$-\frac{1}{96}u^3 + C.$$

Substituting back $u = \tan(\theta)$ we get

$$-\frac{1}{96}\tan^3(\theta) + C.$$

In terms of x, $\tan(\theta) = \frac{\sqrt{4-x^2}}{x}$, so we get

$$-\frac{(4-x^2)^{3/2}}{96x^3} + C.$$

5. (15 points)

Compute the following integral:

$$\int \frac{dx}{(x-2)^2(x+1)}$$

Answer:

We will use partial fractions:

$$\frac{1}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$
$$1 = A(x-2)(x+1) + B(x+1) + C(x-2)^2$$

Using the Heaviside Method, we set x equal to the roots:

$x = 2 : 1 = B(2+1) \to B = \frac{1}{3}$.
$x = -1 : 1 = C(-1-2)^2 \to C = \frac{1}{9}$.

To find A, we need to match the coefficients of x monomials: $1 = A(x^2 - x - 2) + \frac{1}{3}(x+1) + \frac{1}{9}(x^2 - 4x + 4)$.

The coefficient of x^2 on the left side is 0, and on the right side it is $A + \frac{1}{9}$. So $0 = A + \frac{1}{9} \rightarrow A = -\frac{1}{9}$.

Therefore,

$$\begin{aligned} & \int \frac{dx}{(x-2)^2(x+1)} \\ = & \int \frac{-dx}{9(x-2)} + \frac{dx}{3(x-2)^2} + \frac{dx}{9(x+1)} \\ = & -\frac{1}{9}\ln|x-2| - \frac{1}{3(x-2)} + \frac{1}{9}\ln|x+1| + C. \end{aligned}$$

6. (20 points)

Let $f(x)$ be a function defined on an interval $[a, b]$ such that $f(1) = 0$. Suppose that the arc length of the curve $y = f(x)$ on that interval is given by the formula

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx\ .$$

(a) (10 points) Find a function $y = f(x)$ that satisfies the given arc length expression.

Answer:

Using that $L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\, dx$, we find that $[a, b] = [1, 4]$. From this formula, we could find a function that satisfies $(f'(x))^2 = \dfrac{1}{4x}$, or equivalently, $f'(x) = \pm\dfrac{1}{2\sqrt{x}}$. By integrating we obtain that

$$f(x) = \pm \int \frac{1}{2\sqrt{x}}\, dx = \pm\sqrt{x} + C.$$

We have found two valid functions. Namely, $f(x) = \sqrt{x} + C$ or $f(x) = -\sqrt{x} + C$. Given that $f(1) = 0$, we get that in the first case, $f(x) = \sqrt{x} - 1$ or $f(x) = -\sqrt{x} + 1$ in the second case.

(b) (10 points) Write **but do not evaluate** an integral that represents the surface area of the object obtained by revolving the curve $y = f(x)$ about the y-axis.

Answer:

The radius of revolution is given by $r = x$, and the ds is found in the given arc length expression. Therefore,

$$S = 2\pi \int_1^4 x\,\sqrt{1 + \frac{1}{4x}}\,dx.$$

7. (15 points)

The parametric curve

$$\begin{cases} x = t^2 \\ y = t^3 - 2t \end{cases}$$

crosses itself at precisely one point. Find the **slopes** of the **two** tangent lines to the curve at this point. **Simplify your answers as much as possible**.

Answer:

Notice that $x = 2$ for $t = \pm\sqrt{2}$, and in both cases, $y = t^3 - 2t = t(t^2 - 2) = 0$. Therefore, the point $(2, 0)$ on the curve arises from two values of the parameter, $t = \pm\sqrt{2}$. Since

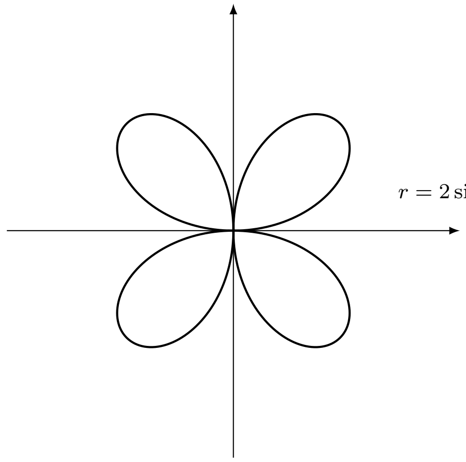
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 2}{2t},$$

the slope of the tangent when $t = \pm\sqrt{2}$ is

$$\frac{6 - 2}{2(\pm\sqrt{2})} = \pm\sqrt{2}$$

Part B

8. (20 points) Consider the polar curve $r = 2\sin(2\theta)$, drawn below.



(a) (10 points) Compute the area inside one leaf of the curve.

Answer:

Using the area formula, we compute

$$\int_0^{\pi/2} \frac{1}{2}\left[(2\sin(2\theta))^2\right] d\theta$$

Using a double angle formula and simplifying this equals

$$\int_0^{\pi/2} [1 - \cos(4\theta)]\, d\theta$$
$$= \frac{\pi}{2}.$$

(b) (10 points) Set up, **but do not evaluate** an integral that computes the arclength of one leaf of the curve.

Answer:

The arclength formula is

$$\int_0^{\pi/2} \sqrt{r^2 + [r']^2} d\theta$$
$$= \int_0^{\pi/2} \sqrt{4\sin^2(2\theta) + 16\cos^2(2\theta)} d\theta$$

9. (20 points)

For this problem, make sure to show all your work: for all the convergence tests that you use state their conditions and explain why they are satisfied.

(a) (10 points) Determine if the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n)}{n^{2k}}$$

is absolutely convergent, conditionally convergent, or divergent when k is a positive integer. Repeat the same question for when $k = 0$.

Answer:

If $k = 0$, then we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\ \sin(n)$$

Let

$$c_n = (-1)^{n-1}\sin(n)$$

then $\lim_{n\to\infty} c_n$ does not exist as it oscillates between -1 and 1; hence, the series is divergent by the Divergence test.

If k is a positive integer then $2k \geq 2$. So,

$$0 \leq \left|(-1)^{n-1}\frac{\sin(n)}{n^{2k}}\right| \leq \frac{1}{n^{2k}} \leq \frac{1}{n^2}$$

because $|\sin(n)| \leq 1$. So the series is absolutely convergent by the Comparison Test since $\sum \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.

Note that in this question neither Alternating Series Test nor Limit Comparison Test works.

(b) (10 points) Let $\{p_n\}$ be the sequence $\{2, 3, 5, 7, 11, 13, \ldots\}$ of all prime numbers (a particular sequence consisting of positive integers). Take as given that

$$\lim_{n\to\infty}\frac{n\ln(n)}{p_n} = 1.$$

Determine if the sum

$$\sum_{n=1}^{\infty}\frac{1}{p_n}$$

of the reciprocals of all prime numbers is convergent or divergent. **Hint:** Use the Limit Comparison Test with a series of the form $\sum_{n=2}^{\infty} b_n$ followed by the Integral Test, justifying all your reasoning.

Answer:

We will apply the Limit Comparison Theorem. Let $a_n = \frac{1}{p_n}$. This is a sequence of positive terms because the reciprocals of prime numbers are positive since primes are positive. To apply the LCT, we need to have another positive sequence b_n and we should know the value of the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{p_n}}{b_n}$$

The fact in the question gives us that if

$$b_n = \frac{1}{n \ln n}, \qquad n \geq 2$$

then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n \ln n}{p_n} = 1$$

Note also that b_n is a sequence with positive terms. As

$$L = 1$$

is a finite non-zero number, by the Limit Comparison Test, the convergence behavior of the series $\sum a_n$ is the same as that of

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

(the starting index is 2 as suggested, the starting index does not affect the convergence behavior and it can't be 1 as the denominator would be 0.)

To see if

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

converges we will apply the Integral Test. The test is applicable because clearly the function $f(x) = (1/x \ln(x))$ is continuous, positive and decreasing on $[2, \infty)$. A brief calculation shows that

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx$$

diverges.

To finalize, the series $\sum b_n$ diverges by the Integral Test and the main series $\sum a_n$ diverges by the Limit Comparison Test.

10. (20 points)

(a) (10 points) Find a power series expansion, centered at $x = 2$, for the function

$$f(x) = \frac{1}{3-x}.$$

Answer:

As $3 - x = 1 - (x - 2)$, we have, for x such that $|x - 2| < 1$:

$$\frac{1}{3-x} = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty}(x-2)^n.$$

(b) (10 points) Use your answer from (a) to find a power series expansion, centered at $x = 2$, for the function $g(x) = \frac{1}{(3-x)^2}$.

Answer:

As $g(x) = f'(x)$ (using the notation of the previous part), we can differentiate the series from (a) term-by-term to obtain a power series representation of $g(x)$. Hence,

$$g(x) = \sum_{n=1}^{\infty} n(x-2)^{n-1}.$$

11. (20 points)

(a) (10 points) Find the Taylor series of the function $f(x) = \ln(x)$, centered at $x = 2$ (i.e., in powers of $x - 2$).

Answer:

We have to compute all the derivatives of $f(x) = \ln x$, evaluated at $x = 2$.

We have $f^0(x) = \ln(x)$, $f^0(2) = \ln 2$

$f^1(x) = 1/x$, $f^1(2) = 1/2$

$f^2(x) = -1/x^2$, $f^2(2) = 1i2^2$,

$f^3(x) = 2/x^3$, $f^3(2) = 2/2^3$,

$f^4(x) = -2 \cdot 3/x^4$, $f^4(2) = -2 \cdot 3/2^4$,

$f^5(x) = 2 \cdot 3 \cdot 4/x^5$, $f^4(2) = 2 \cdot 3 \cdot 4/2^5$,

So for $n \geq 1$, $f^n(x) = (-1)^{n+1}(n-1)!/x^n$, $f^n(2) = (-1)^{n+1}(n-1)!/2^n5$,

So $\ln(x) = f(2) + \sum_{n=1}^{\infty}(-1)^{n+1}/n! \cdot (n-1)!/2^n(x-2)^n$, which simplifies to

$$\ln(x) = f(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} \cdot (x-2)^n$$

Using the ratio test, we get that the radius of conversion is $R = 2$. So the endpoints of the interval of conversion are $2 - 2 = 0$ and $-2 - 2 = -4$. When $x = 0$, the series becomes the negative of the divergent harmonic series, so the series doeesn't converge at the endpoint 0. When $x = 4$, the series becomes the negative of the alternating harmonic series, which converges by the alternating series test. So the interval of convergence is $(0, 4]$.

(b) (10 points) Find the radius of convergence and interval of convergence of your series from part (a).

Answer:

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Second scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Third scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

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And even more scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.