Math 162: Calculus IIA

Final Exam

May 2, 2022

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Indicate your instructor with a check in the box:

Sergio Chaves	MW 10:25 - 11:40 AM	
Arda Demirhan	MW 12:30 - 1:45 PM	
Bogdan Krstić	MW 2 - 3:15 PM	
Saul Lubkin	TR 9:40 - 10:55 PM	
Charles Wolf	MW 3:25 - 4:40 PM	

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

- **The presence of calculators, cell phones and other electronic devices at this exam is strictly forbidden and WILL BE TREATED AS AN ACADEMIC HONESTY VIOLATION.**
- **Show your work and justify your answers. Put your answers in the space provided at the bottom of each page or half page. SIMPLIFY YOUR ANSWERS AS MUCH AS POSSIBLE.**
- **Part A (problems 1–7) covers the same material as the two midterms, and Part B (problems 8–11) covers additional material. Letter grades will be computed for the two parts separately. Part B will count for 20% of your course grade. Part A will count for at least 10% of your course grade. If your letter grade on part A is better than your lowest midterm letter exam grade, then it will replace that midterm exam grade and count for 30% of your course grade.**
- **You are responsible for checking that this exam has all 27 pages.**

Handy dandy formulas

Integration by parts formula: $\int u\,dv = uv - \int v\,du$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Area of surface of revolution in rectangular coordinates for $y = f(x)$ with $a \le x \le b$

- about the x-axis: $S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$
- about the y-axis: $S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$

Polar coordinates

$$r = \sqrt{x^2+y^2} \qquad \theta = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi + \arctan(y/x) & \text{for } x < 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{for } (x,y) = (0,0) \end{cases}$$

$$x = r\cos\theta, y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

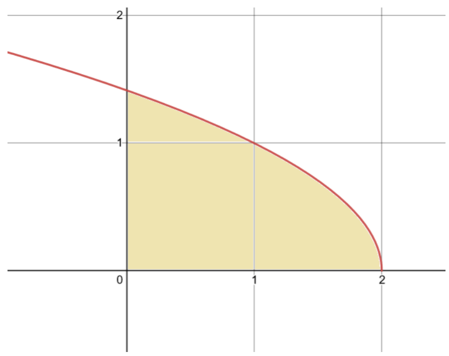
Infinite series formulas

- The *Maclaurin series* for $f(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
- The *Taylor series* for $f(x)$ at a is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.
- The *mth Taylor polynomial* is $T_m(x) = \displaystyle\sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!} (x-a)^n$,
- The *mth Taylor remainder* is $R_m(x) = f(x) - T_m(x)$
- Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then $|R_m(x)| \leq \dfrac{(x-a)^{n+1}M}{(n+1)!}$.

Part A

1. (20 points)

Consider the region in the first quadrant enclosed by the curve $y = \sqrt{2-x}$ and the lines $x = 0$ and $y = 0$. A sketch of the region is shown in the following graph:



(a) (10 points)

Use the **shell method** to write an integral that represents the volume of the solid obtained by revolving the region about the y-axis. **Do not evaluate the integral!**.

ANSWER:

(b) (10 points) Use the **disk/washer method** to write an integral that represents the volume of the solid obtained by revolving the region about the line $y = 2$. **Do not evaluate the integral!**.

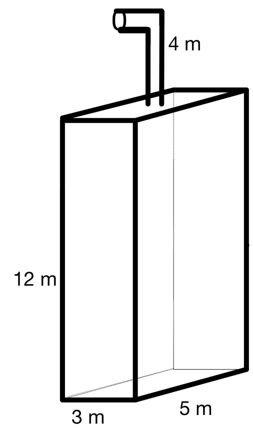
ANSWER:

2. (20 points) A rectangular water tank that is 5 meters long, 3 meters wide, and 12 meters deep is full of water. Take the density of water as 1000 kg/m^3 and approximate the acceleration due to gravity as 10 m/s^2.

(a) (10 points) Set up **but do not evaluate** the integral expression for the work needed to pump all of the water out of the tank.

ANSWER:

(b) (10 points) Set up **but do not evaluate** the integral expression for the work needed to pump one fourth of the water out with a pipe that is 4 m above the top of the tank.



ANSWER:

3. (15 points) Compute the following integral:

$$\int 6\arctan\left(\frac{8}{w}\right)\, dw$$

ANSWER:

4. (15 points)

Compute the following integral:

$$\int \frac{\sqrt{4-x^2}}{x^4}\,dx$$

ANSWER:

5. (15 points)

Compute the following integral:

$$\int \frac{dx}{(x-2)^2(x+1)}$$

ANSWER:

6. (20 points)

Let $f(x)$ be a function defined on an interval $[a, b]$ such that $f(1) = 0$. Suppose that the arc length of the curve $y = f(x)$ on that interval is given by the formula

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx\ .$$

(a) (10 points) Find a function $y = f(x)$ that satisfies the given arc length expression.

ANSWER:

(b) (10 points) Write **but do not evaluate** an integral that represents the surface area of the object obtained by revolving the curve $y = f(x)$ about the y-axis.

ANSWER:

7. (15 points)

The parametric curve

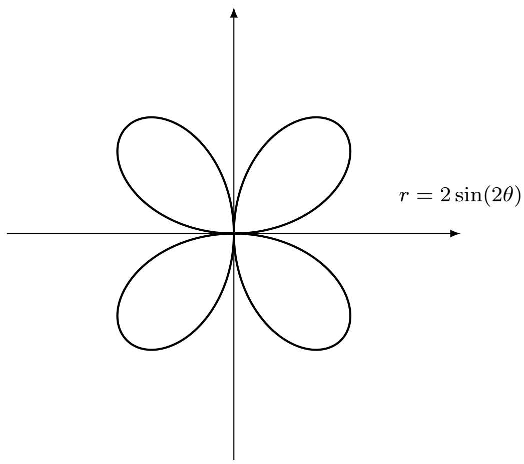
$$\begin{cases} x = t^2 \\ y = t^3 - 2t \end{cases}$$

crosses itself at precisely one point. Find the **slopes** of the **two** tangent lines to the curve at this point. **Simplify your answers as much as possible**.

ANSWER:

Part B

8. (20 points) Consider the polar curve $r = 2\sin(2\theta)$, drawn below.



(a) (10 points) Compute the area inside one leaf of the curve.

ANSWER:

(b) (10 points) Set up, **but do not evaluate** an integral that computes the arclength of one leaf of the curve.

ANSWER:

9. (20 points)

For this problem, make sure to show all your work: for all the convergence tests that you use state their conditions and explain why they are satisfied.

(a) (10 points) Determine if the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n)}{n^{2k}}$$

is absolutely convergent, conditionally convergent, or divergent when k is a positive integer. Repeat the same question for when $k = 0$.

$k \neq 0$ ANSWER:

$k = 0$ ANSWER:

(b) (10 points) Let $\{p_n\}$ be the sequence $\{2, 3, 5, 7, 11, 13, \dots\}$ of all prime numbers (a particular sequence consisting of positive integers). Take as given that

$$\lim_{n\to\infty} \frac{n\ln(n)}{p_n} = 1.$$

Determine if the sum

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

of the reciprocals of all prime numbers is convergent or divergent. **Hint:** Use the Limit Comparison Test with a series of the form $\sum_{n=2}^{\infty} b_n$ followed by the Integral Test, justifying all your reasoning.

ANSWER:

10. (20 points)

(a) (10 points) Find a power series expansion, centered at $x = 2$, for the function

$$f(x) = \frac{1}{3 - x}.$$

ANSWER:

(b) (10 points) Use your answer from (a) to find a power series expansion, centered at $x = 2$, for the function $g(x) = \dfrac{1}{(3 - x)^2}$.

ANSWER:

11. (20 points)

(a) (10 points) Find the Taylor series of the function $f(x) = \ln(x)$, centered at $x = 2$ (i.e., in powers of $x - 2$).

ANSWER:

(b) (10 points) Find the radius of convergence and interval of convergence of your series from part (a).

ANSWER:

This is scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Second scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

Third scratch paper page. If you use it to work on a problem, please indicate so on the page where that problem occurs.

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And even more scratch paper. If you use it to work on a problem, please indicate so on the page where that problem occurs.