MTH161 Workshop 7: implicit differentiation, derivatives of logs, logarithmic differentiation, rates of change and 1-D motion

1. (a) Check that the derivatives of $\ln(x)$ and $\ln(2x)$ are the same. Can you explain why these two functions should have the same derivative?

(b) Use implicit differentiation to show that $\frac{d}{dx}\log_a(x) = \frac{1}{x \ln a}$. **Hint:** If $y = \log_a(x)$, then $a^y = x$.

(c) Explain the difference between $\cos^{-1} x$ and $(\cos x)^{-1}$.

(d) Use implicit differentiation to show that $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$. Also find $\frac{d}{dx}((\cos x)^{-1})$.

2. Let h, k, and r be constants, and consider the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

(a) Help your scribe sketch the graph of the equation. Label your graph — in particular, show how the constants h, k, and r are relevant to the graph.

(b) By just looking at the graph, determine at which points the tangent line is horizontal and at which points the tangent line is vertical.

(c) Now, find $\frac{dy}{dx}$ and verify your answers to part (b).

3. Consider

$$f(x) = \frac{e^x \sqrt{x}(x-2)^3}{\sqrt[4]{(x^2+1)^3}\ln(x)}$$

(a) Use logarithmic differentiation to find $f'(x)$.

(b) Find $f'(2)$.

4. Consider the equation $x^y = y^x$. Find $\dfrac{dy}{dx}$ in terms of x and y.

5. An object is traveling along the x-axis. Its position at time t seconds ($t \geq 0$) is given by

$$x(t) = t^3 - 9t^2 + 24t \text{ ft.}$$

With your group, answer the following questions about the motion of the object.

(a) What is the velocity at time t?
(b) When is the object at rest?
(c) When is the particle moving in the *forward* direction? In the *backward* direction?
(d) What is the acceleration at time t?
(e) When is the particle speeding up? Slowing down?
(**Hint:** The object is speeding up when it's accelerating in the same direction it's moving, and slowing down when it's accelerating in the opposite direction.)
(f) What is the **total** distance the object travels during the first 8 seconds?

6. A model for the spread of a rumor is given by

$$p(t) = \frac{1}{1 + ae^{-kt}},$$

where $p(t)$ is the proportion of the population that knows the rumor at time t. (Here, a and k are positive constants.)

(a) When will half the population have heard the rumor?
(b) What happens to $p(t)$ as $t \to \infty$? What does this mean?
(c) Find $p'(t)$. What does this represent?
(d) What happens to $p'(t)$ as $t \to \infty$? What does this mean?

Challenge Problem:

(a) Show that $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$.

(b) Use part (a) to show $\dfrac{d}{dx}\csc^{-1} x = \dfrac{-1}{x^2\sqrt{1 - 1/x^2}} = \dfrac{-1}{|x|\sqrt{x^2 - 1}}$.

(c) Similarly write $\sec^{-1} x$ in terms of $\cos^{-1}$ and find the derivative of $\sec^{-1} x$.