Math 161

Spring 2022

Worksheet 4

Derivatives and Rates of Change; Derivatives of Powers, Exponentials

Problem Set Instructions: Work through the following problems with your group. You might not finish all of the problems, but be sure to work on all of them together and gain a good idea of how to proceed.

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1. The binomial theorem tells us that for a positive integer n, that $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are called binomial coefficients. As it turns out, the binomial coefficients for n are given by the nth row of Pascal's triangle (if you've never heard of that, Google it! or ask your group or TA). For example, the 3rd row of Pascal's triangle is 1, 3, 3, 1 and these are the coefficients on the terms of $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

   (a) Expand $(x+y)^5$.

   (b) Let $f(x) = x^5$ and use the definition of derivative $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to show $f'(x) = 5x^4$. Hint: Part (a) should help.

2. Using the fact that $\lim_{t\to 0}\dfrac{e^t-1}{t} = 1$, show (using the definition of the derivative) that

$$\frac{d}{dx}e^x = e^x.$$

3. The **vertex** of a parabola is the point at which the tangent line is horizontal. Show that the vertex of the graph of $y = ax^2 + bx + c$ has x-coordinate $-\frac{b}{2a}$.

4. (a) For which values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ at $x = 2$?

   (b) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx + b & \text{if } x > 2 \end{cases}.$$

   Find the values for m and b that make f differentiable everywhere. (You'll need to use the definition of the derivative here.)

5. The temperature T, in degrees Fahrenheit, of a cold potato placed in a hot oven is given by $T = h(t)$, where t is the time in minutes since the potato was put into the oven.

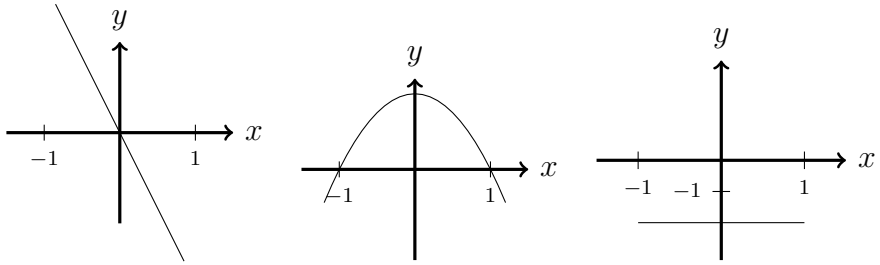
   (a) Discuss with your group what $h(20) = 125$ means in the context of this application.

   (b) Discuss with your group what $h'(20) = 2.3$ means in the context of this application.

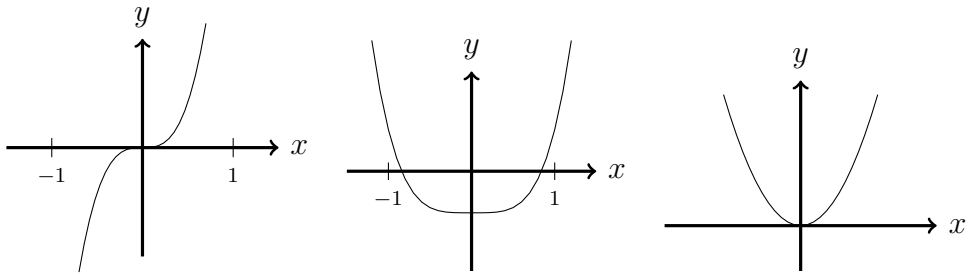
   (c) Use this information to estimate the temperature of the potato 21 minutes after it was put in the oven.

   (d) What do you think happens to $h(t)$ as t goes to infinity? What about $h'(t)$? Discuss with your group.

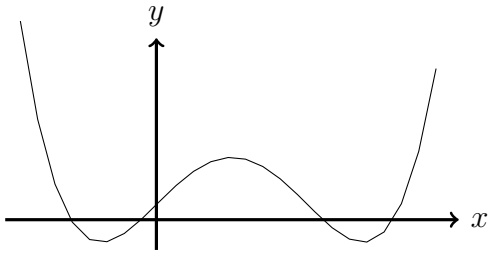
6. (a) Below are the graphs of $f(x)$, $f'(x)$, and $f''(x)$ for some function $f(x)$. Determine which graph belongs to which function.



(b) Below are the graphs of $g(x)$, $g'(x)$, and $g''(x)$ for some function $g(x)$. Determine which graph belongs to which function.



(c) The graph below shows $y = h(x)$ for some function h. Sketch the graph of $h'(x)$.



Challenge 1: Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ -x^2, & \text{if } x \text{ is irrational.} \end{cases}$$

Is $f'(0)$ defined? (Again, you'll need to use the definition of the derivative.)

Challenge 2: Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$. What is the point of tangency for the c you found?