MTH161 Workshop 2: transformations of functions, exponential and logarithmic functions, inverse functions

Problem Set Instructions: Work through the following problems with your group. You might not finish all of the problems, but be sure to work on all of them together and gain a good idea of how to proceed.

Discussion: Discuss the following question(s) with your group.

- If $f(x)$ is a function and a and b are two real numbers, is the equation
$$f(a + b) = f(a) + f(b)$$
always true? If so, explain why. If not, find an example where the equation does not hold.
- Let $f(x)$ be a function with domain D. What can you say about the domain of the function $\dfrac{f(x)}{e^x}$? How about the domain of $f(x) + \ln(x)$?

1. Consider the following questions with your group.
   (a) Sketch the graph of $\cos(x)$
   (b) Starting with the graph of $\sin(x)$, use a series of graph transformations to sketch the graph of $\sin\left(\dfrac{\pi}{2} - x\right)$. How does it compare to the graph of $\cos(x)$?
   (c) By drawing an appropriate right triangle, convince yourselves that if θ is an **acute** angle (i.e. $\theta \in (0, \pi/2)$), then $\cos(\theta) = \sin\left(\dfrac{\pi}{2} - \theta\right)$.
   (d) Use the trigonometric identity $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ to prove that $\cos(\theta) = \sin\left(\dfrac{\pi}{2} - \theta\right)$ for all $\theta \in \mathbb{R}$.

2. Consider the following questions with your group.
   (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
      Hint: Think about what operations you would have to perform on the formula for g to end up with the formula for h.
   (b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

3. With your group, consider the function
$$f(x) = \ln(|x - 4|) - 1.$$
   (a) Find the domain and range of f.
   (b) Sketch the graph of f. **Hint:** Start with the graph of $\ln(x)$

4. With your group:
  (a) Simplify the expression $\cot(\arcsin(z))$.
  (b) Find the exact value of the expressions $\cos(2\sin^{-1}(5/13))$
  (c) Find the exact value of the expressions $\sin(2\sin^{-1}(5/13))$.

5. A typical cup of coffee contains about 100 mg of caffeine, and every hour approximately 16% of the amount of caffeine in the body is metabolized and eliminated.
  (a) After drinking a cup of coffee, determine how much caffeine is in the body after 1 hour. After 2 hours? 5 hours?
  (b) Write C, the amount of caffeine in the body (in mg), as a function of t, the time (in hours) since the coffee was consumed.
  (c) Discuss what happens to the caffeine in the body as time increases. Does this make sense?
  (d) Find the half-life of caffeine, that is, find the time at which exactly half of the initial dose remains in the body.

6. Consider the following function with your group:
$$f(x) = \begin{cases} -x^2 - 1, & \text{if } x < 0; \\ e^x - 2, & \text{if } x \geq 0. \end{cases}$$
  (a) Help your scribe sketch the graph of $f(x)$.
  (b) Discuss with each other why the function is one-to-one and why the range is $\mathbb{R}$. (Use your sketch of the graph.)
  (c) Help your scribe sketch $f^{-1}(x)$ without working out what the function f^{-1} actually is.
  (d) Together, write out $f^{-1}(x)$ as a piecewise-defined function.
      Hint: Find the inverse of each component of $f(x)$, keeping in mind their restricted domains and ranges.